Precoloring co-Meyniel graphs

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Graphs & Combinatorial Optimization group

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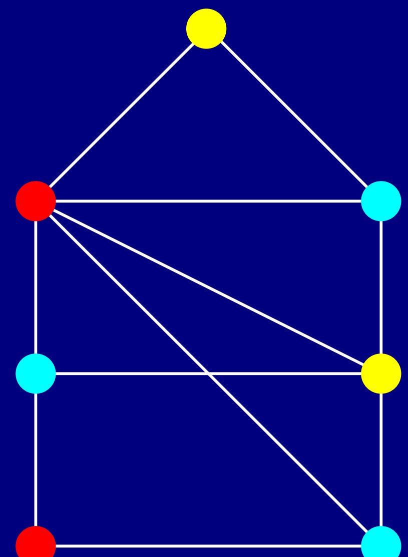

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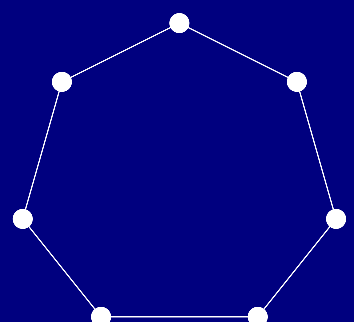

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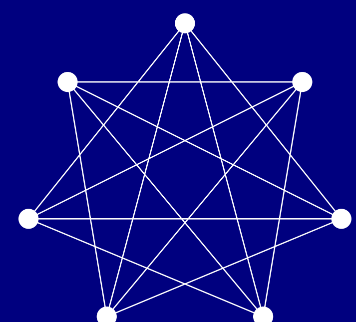

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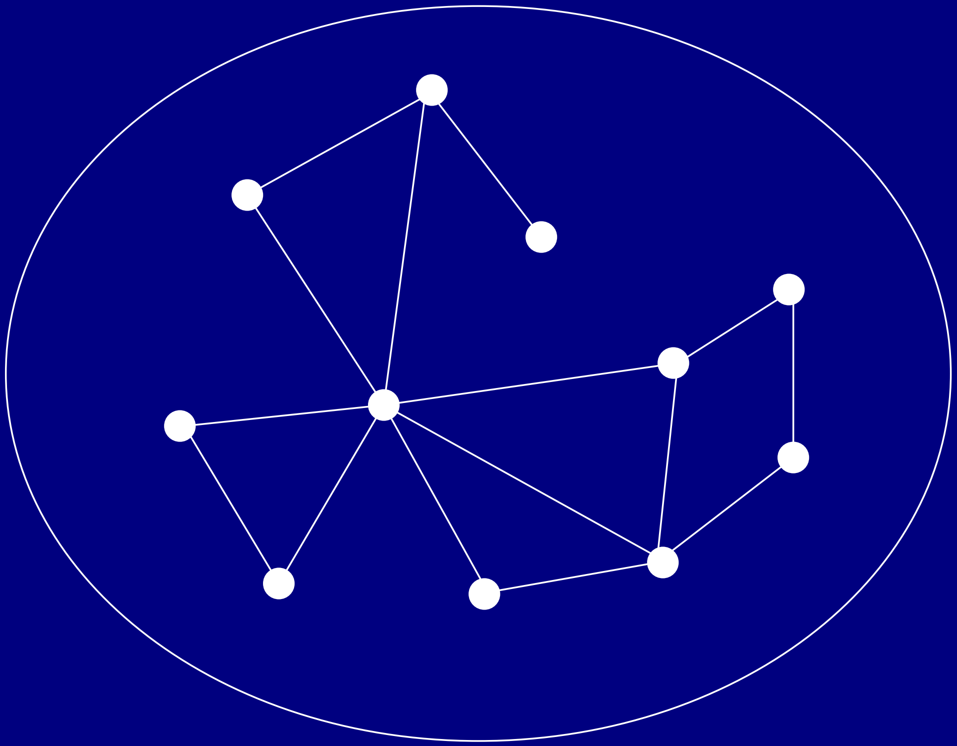
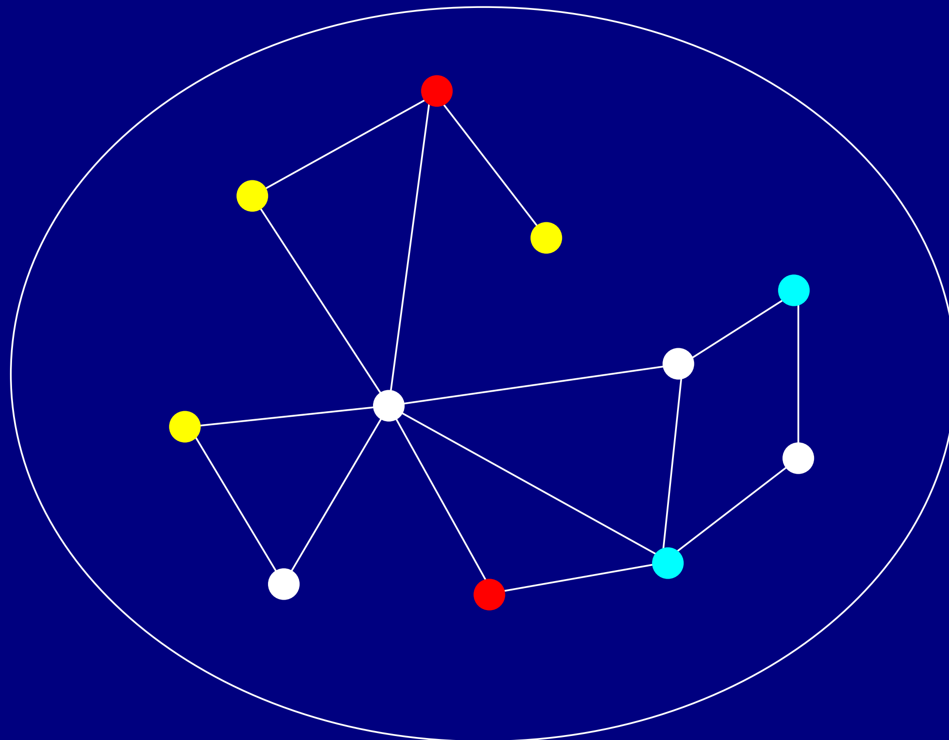
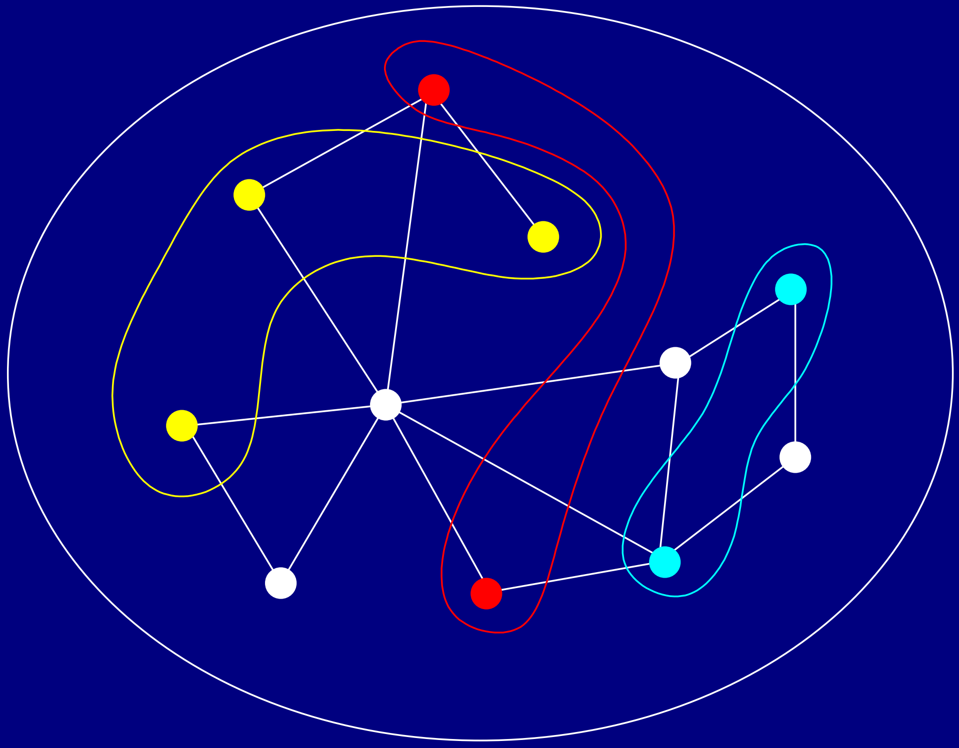
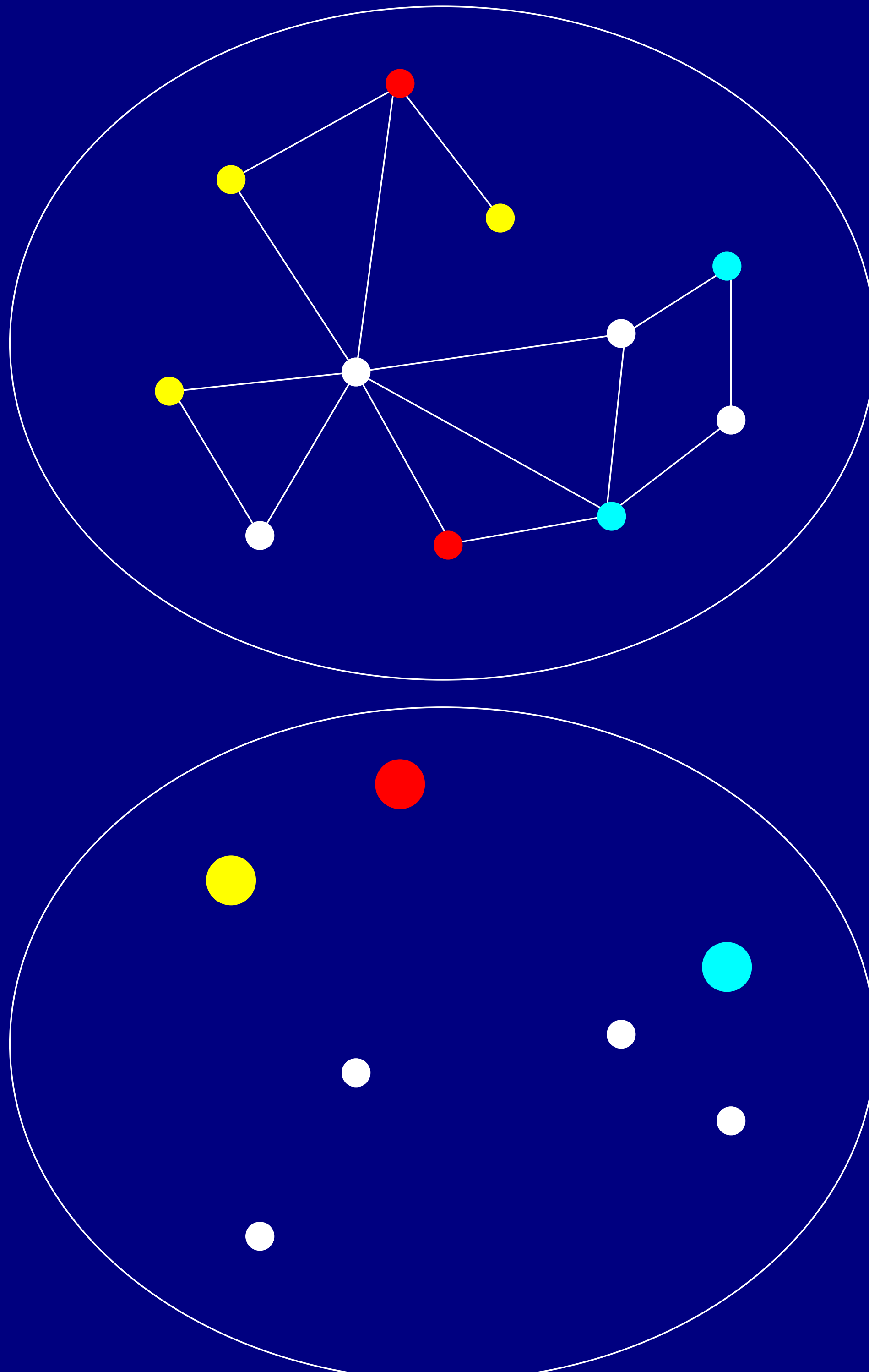
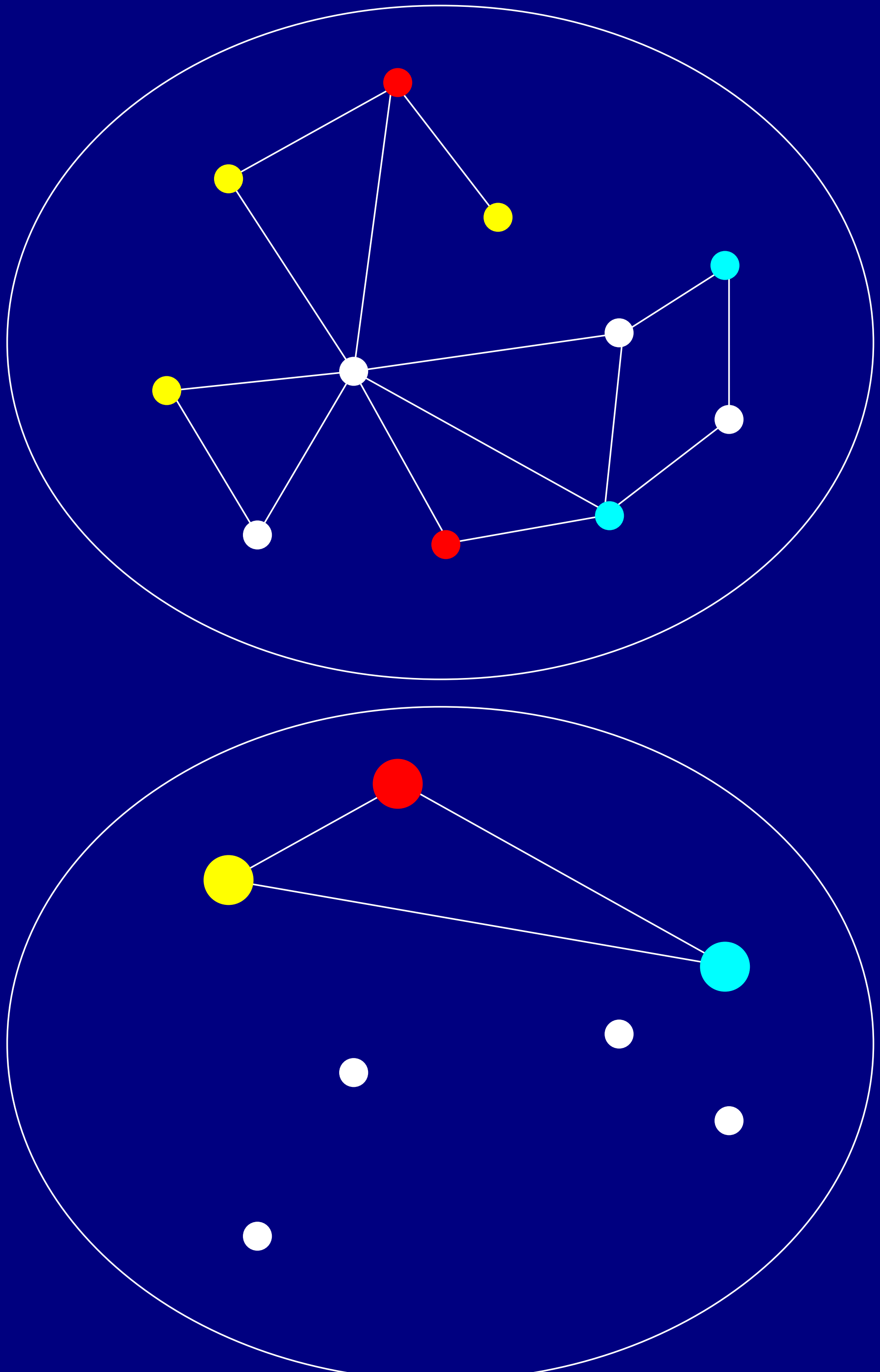
- NP-complete on bipartite graphs, interval graphs, permutation graphs.

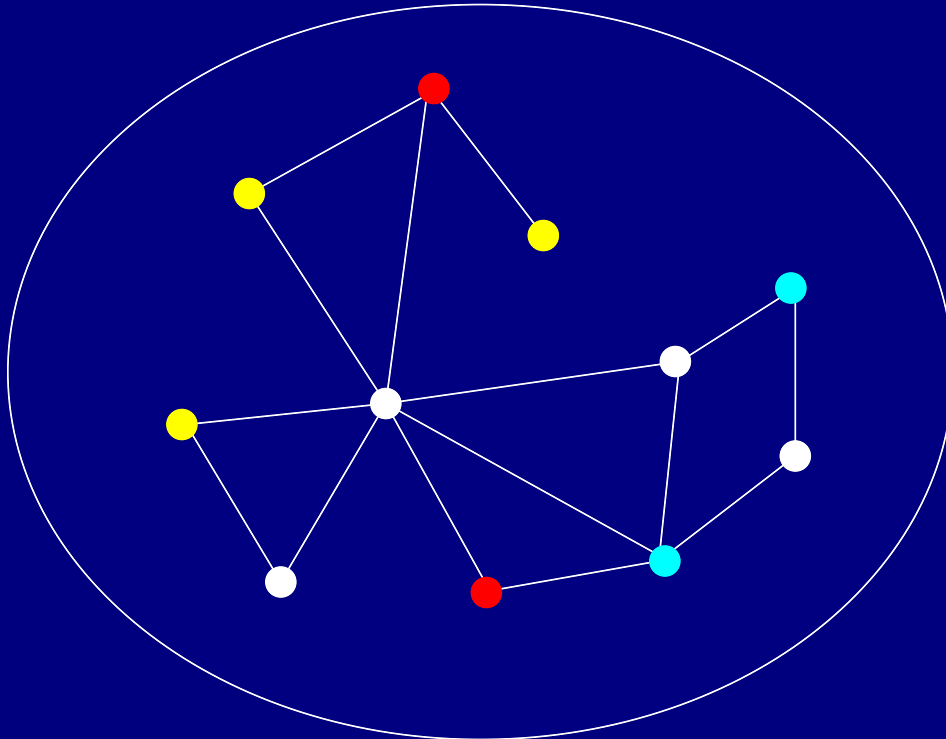
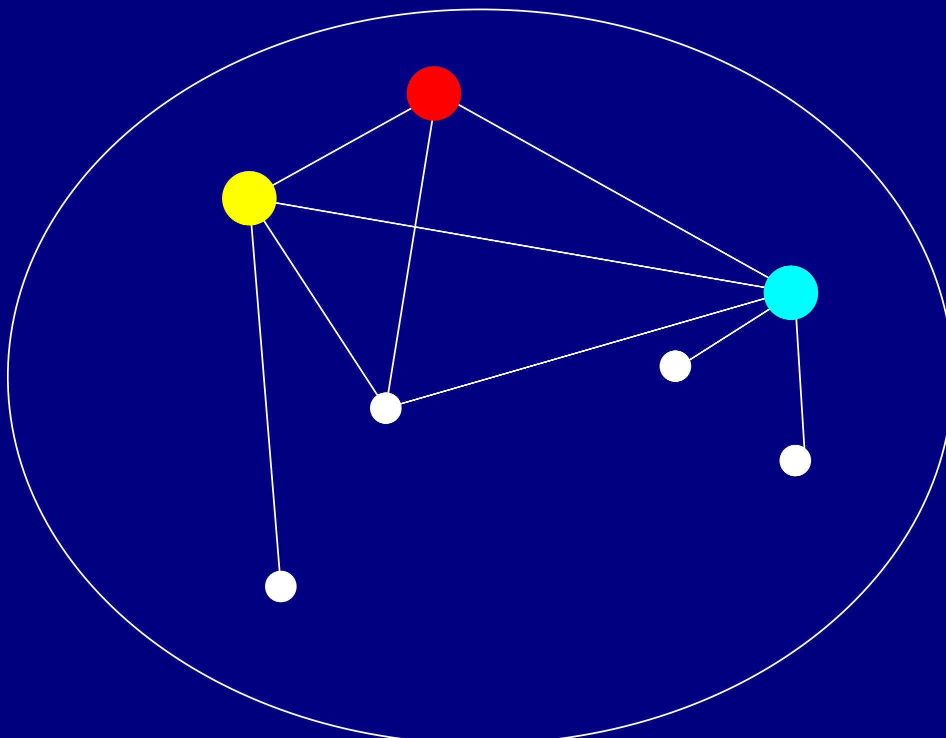


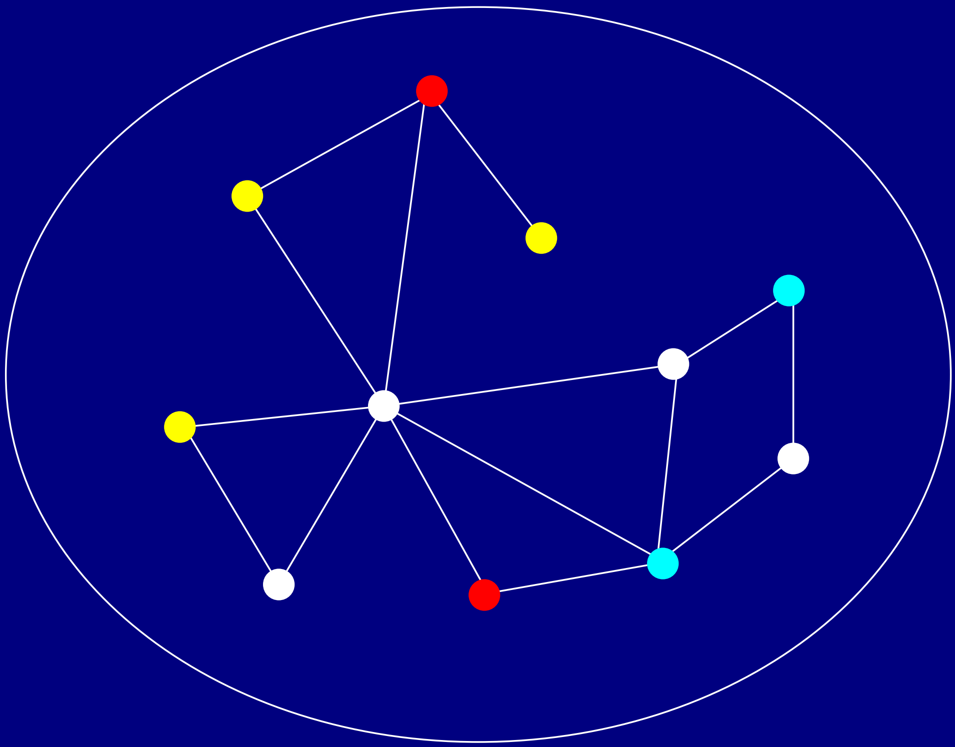


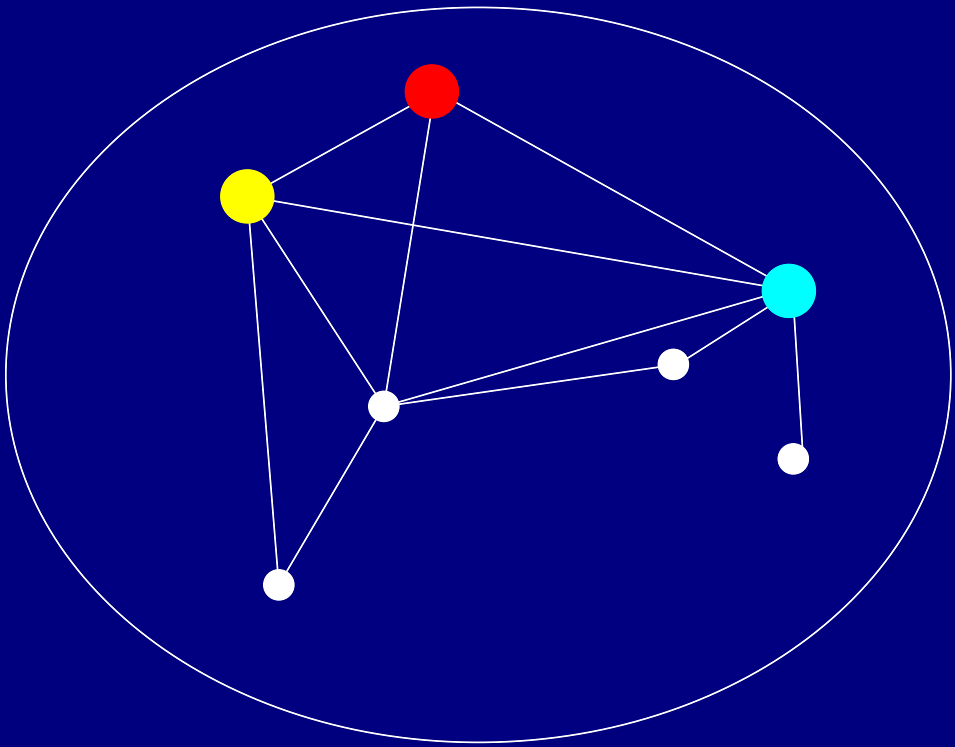


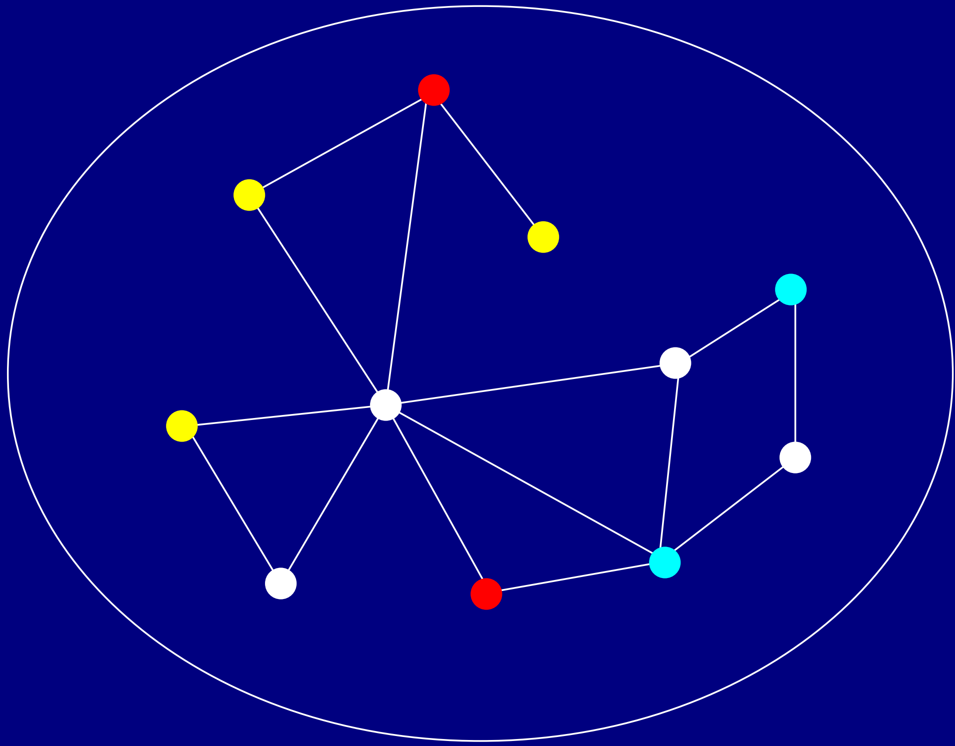


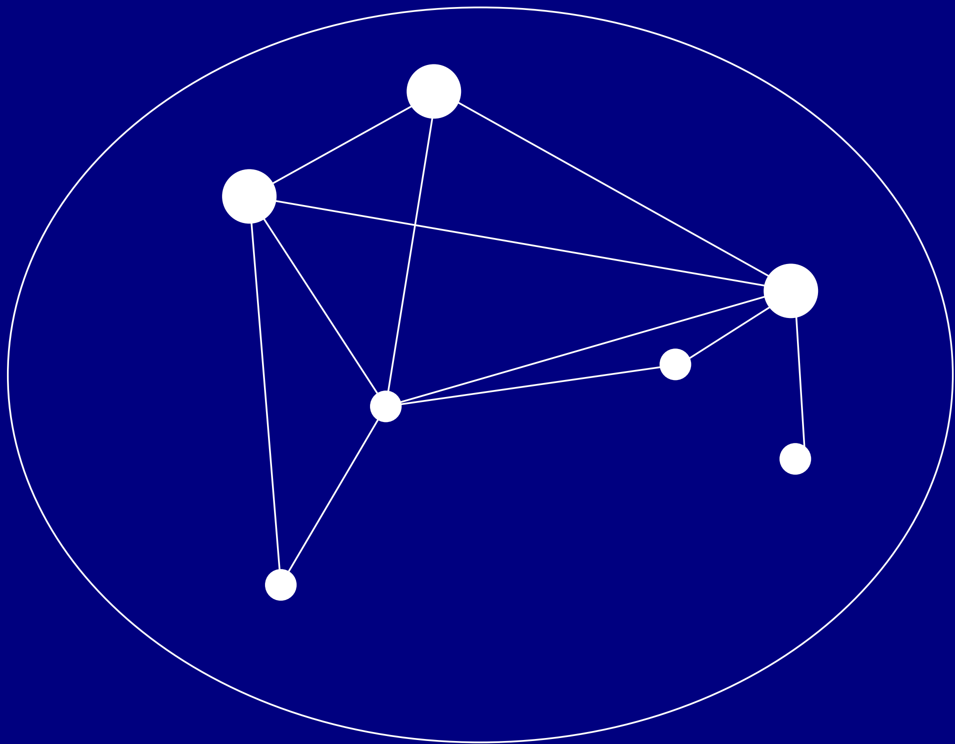










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Which are the PrExt-perfect graphs?

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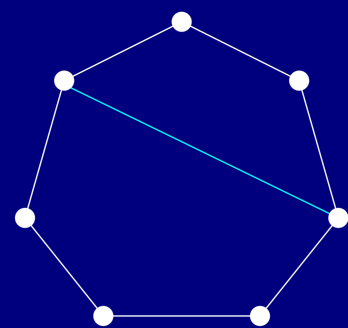
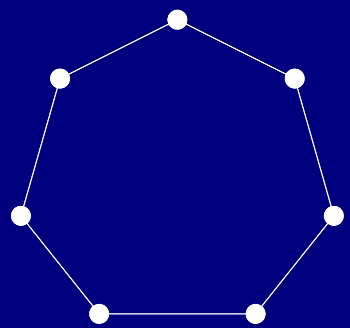


- Meyniel Graphs
  Every odd cycle has at least two chords

  $\Longleftrightarrow$ contains no odd hole and no house

Odd hole



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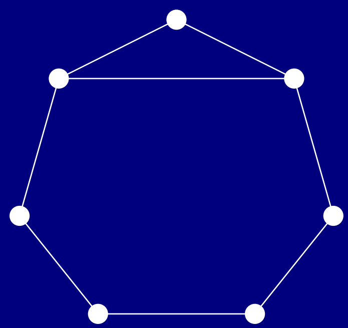

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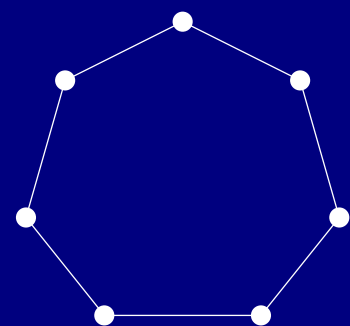

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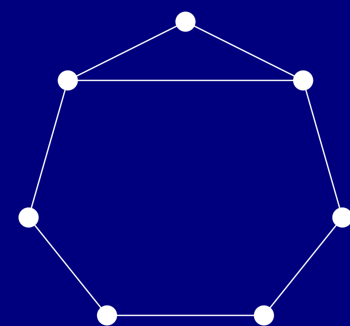
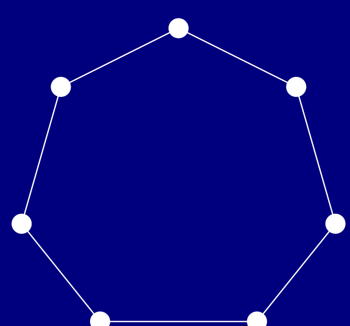


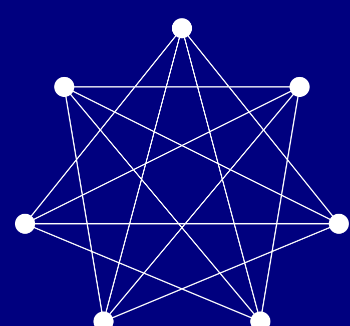
- Artemis Graphs

  Contain no odd hole, no antihole (odd or even), and no prism

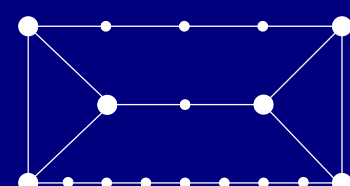
Odd hole



Antihole



Prism



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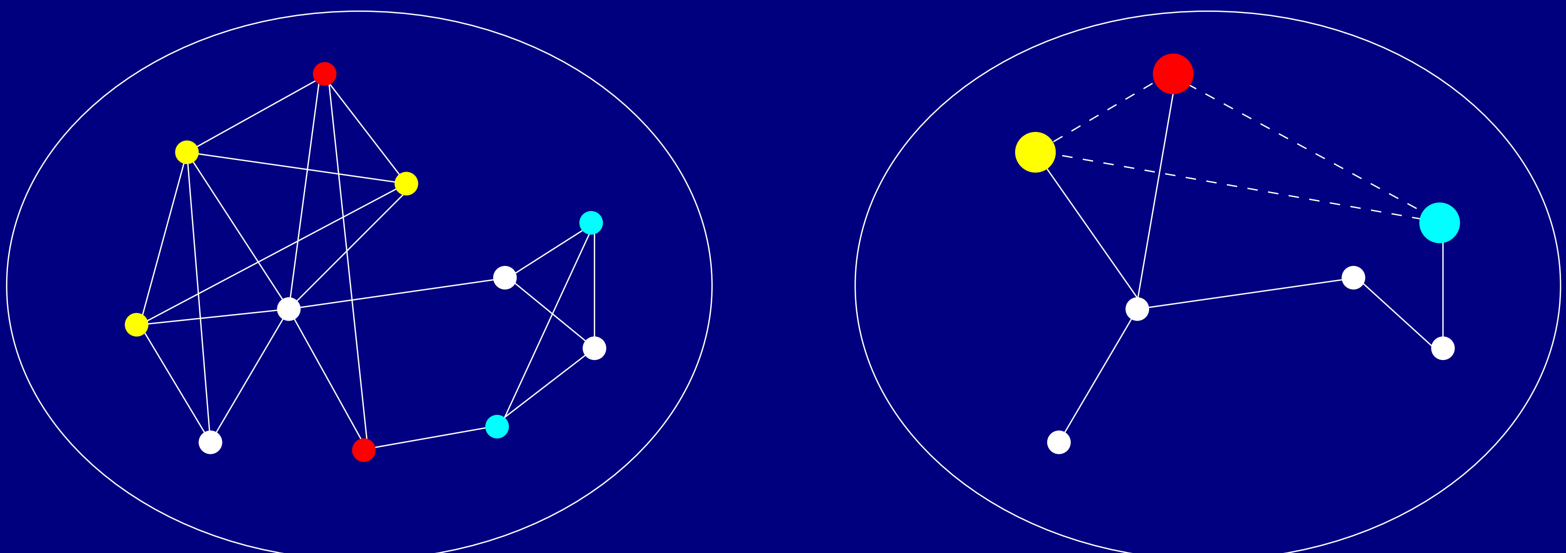

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Generalization of known results

Split graphs, cographs ($= P_4$-free graphs), P_5-free bipartite graphs, complements of bipartite graphs [Hujter, Tuza]

Co-Meyniel graphs with all pre-coloring classes of size 1 [Hertz]

$\forall \mathcal{Q},\ G^{\mathcal{Q}}$ *perfect* $\implies G$ *Meyniel*

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- G contains an odd hole
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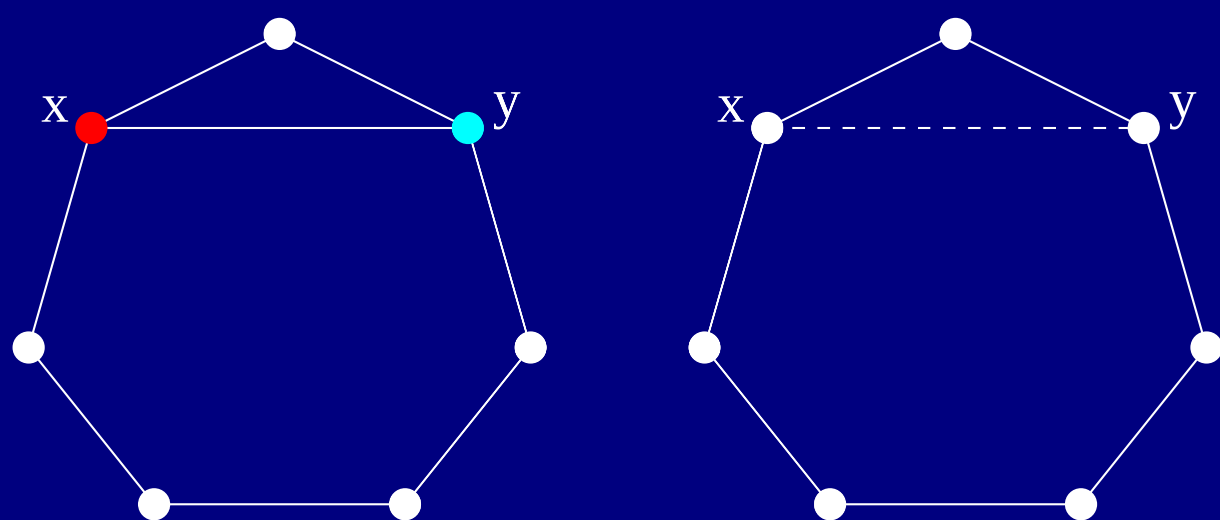

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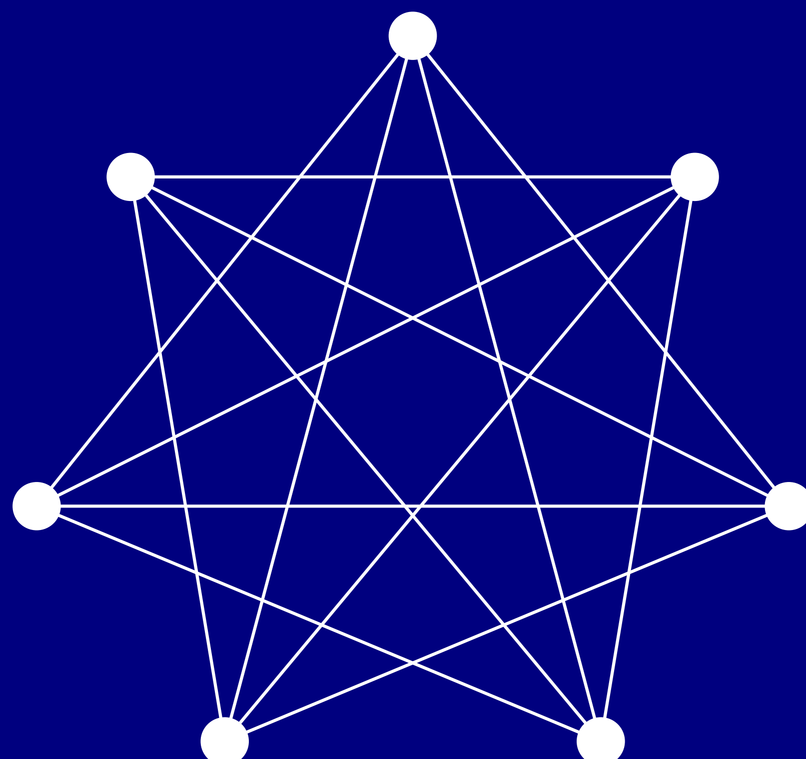

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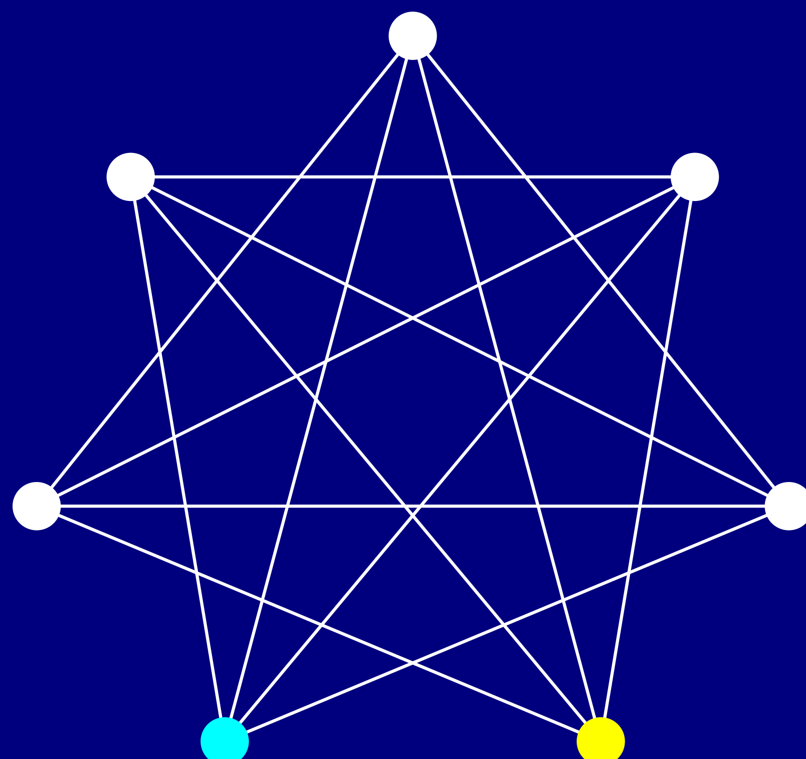

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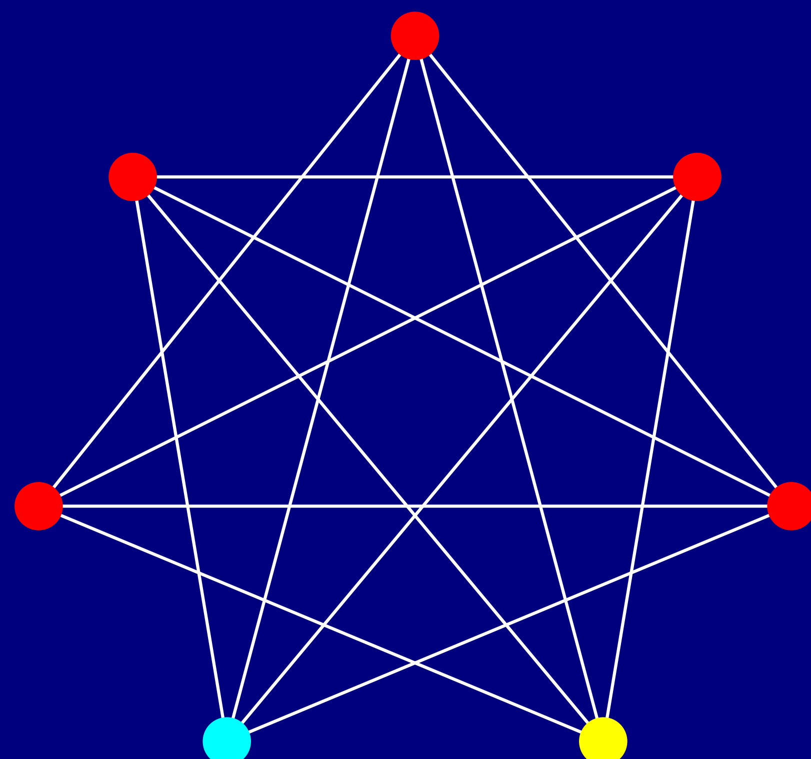

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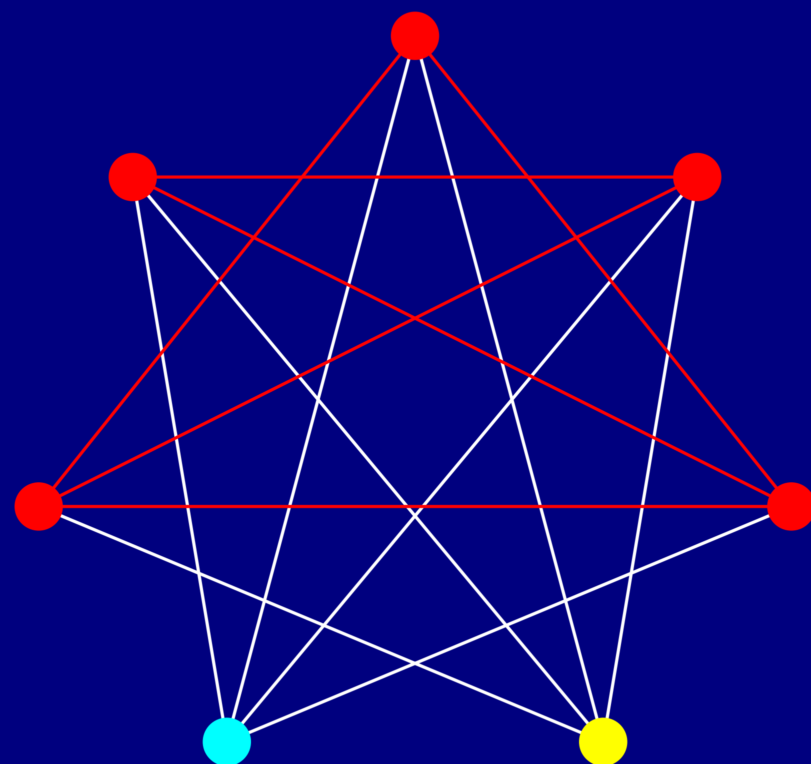

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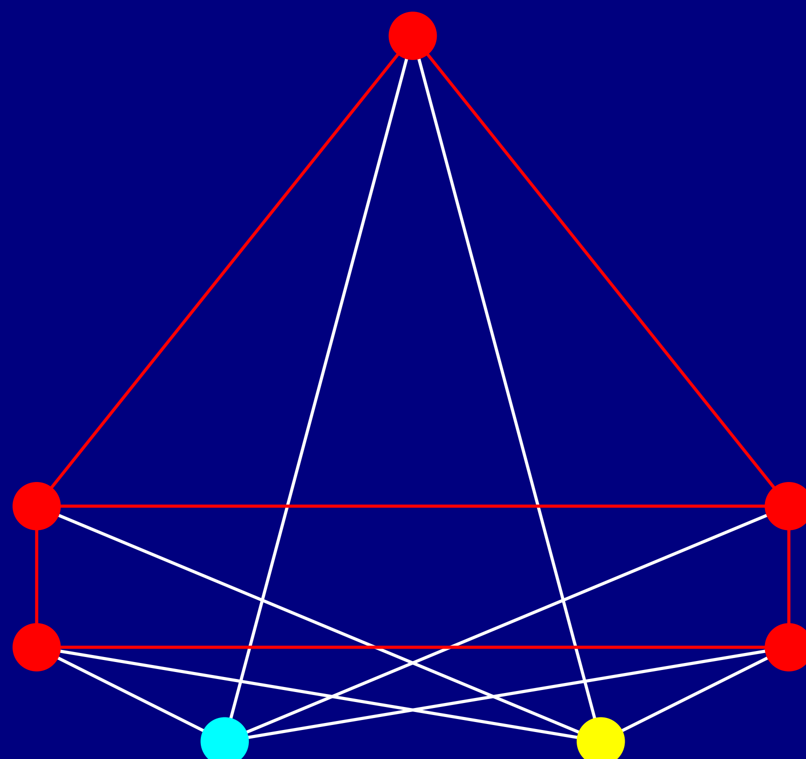

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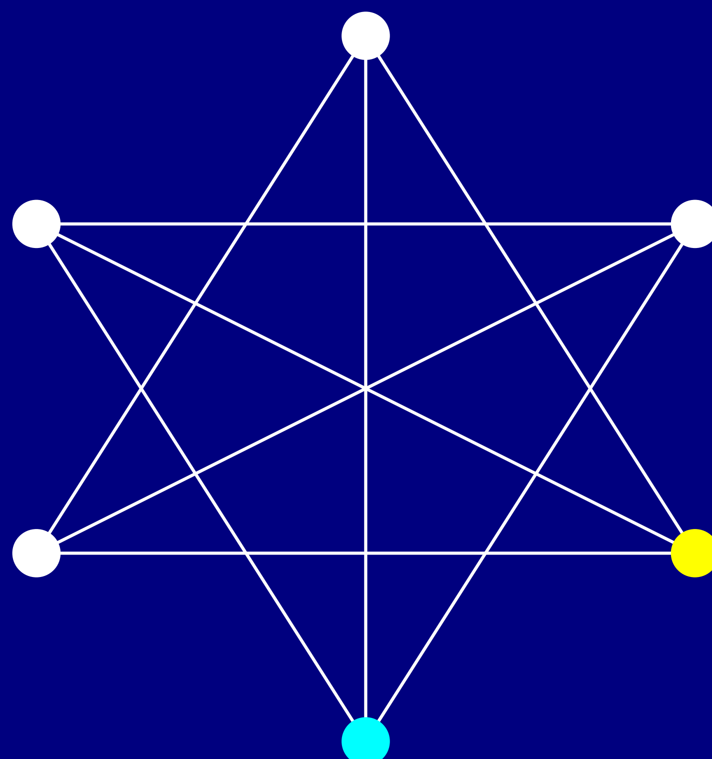

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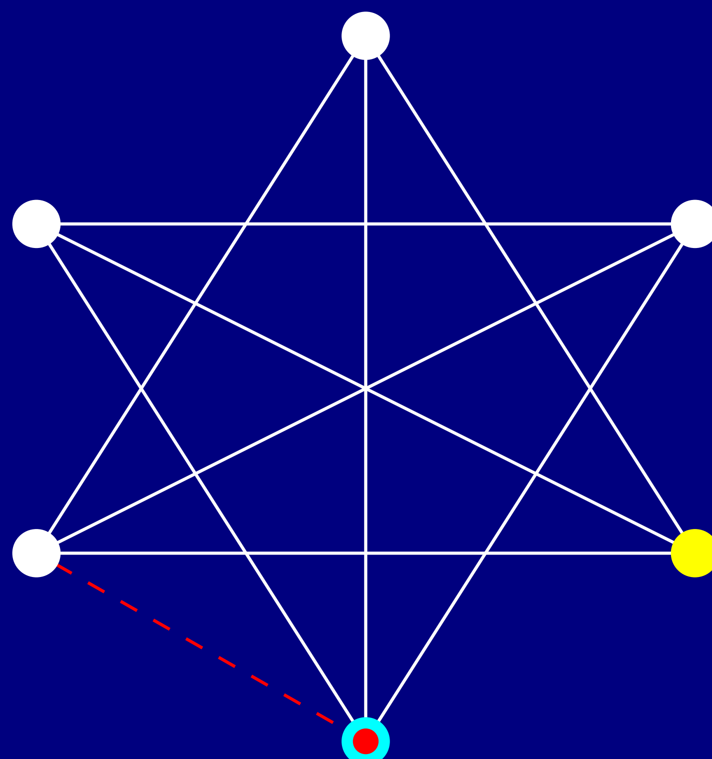

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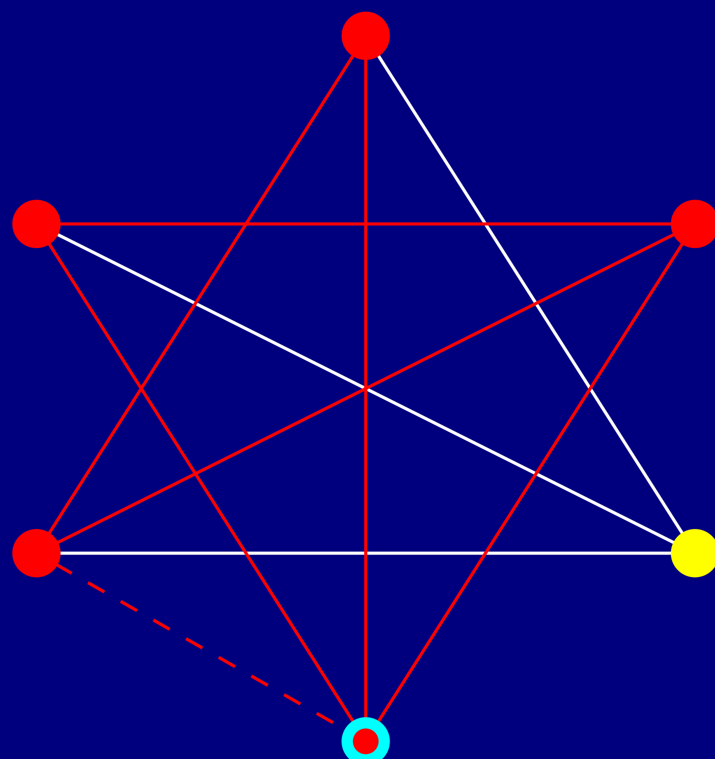

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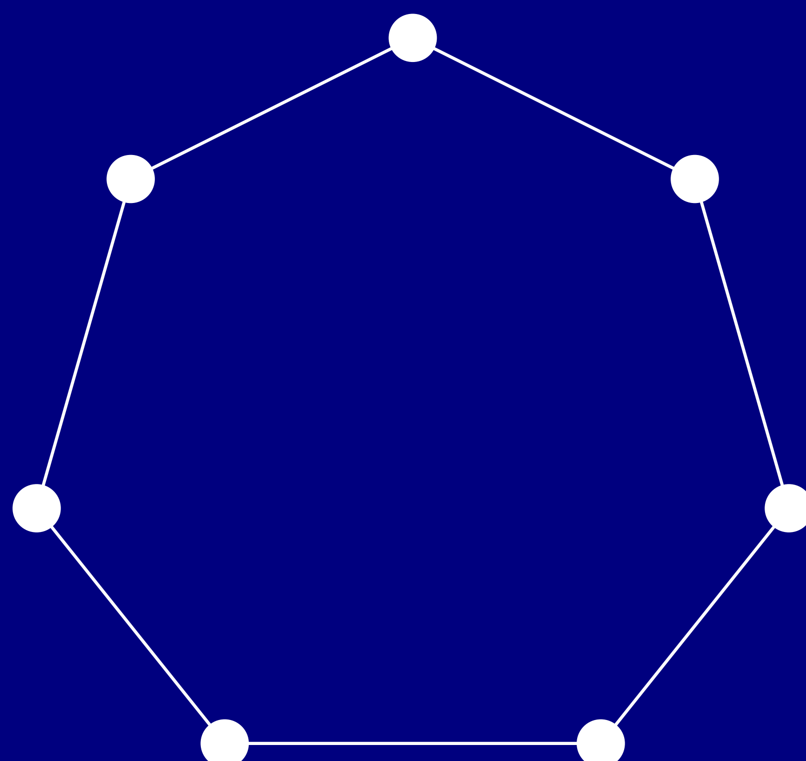

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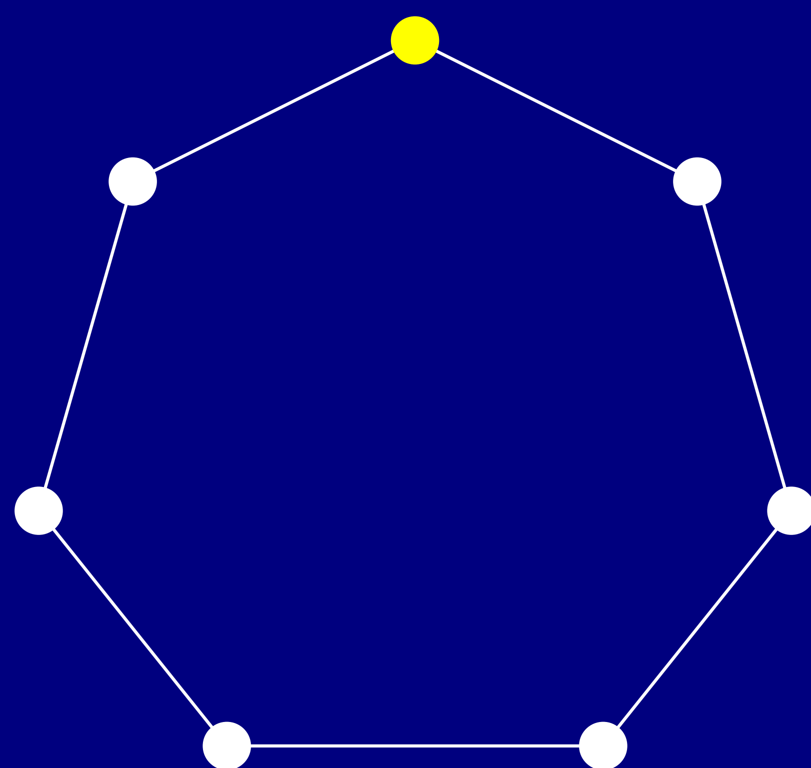

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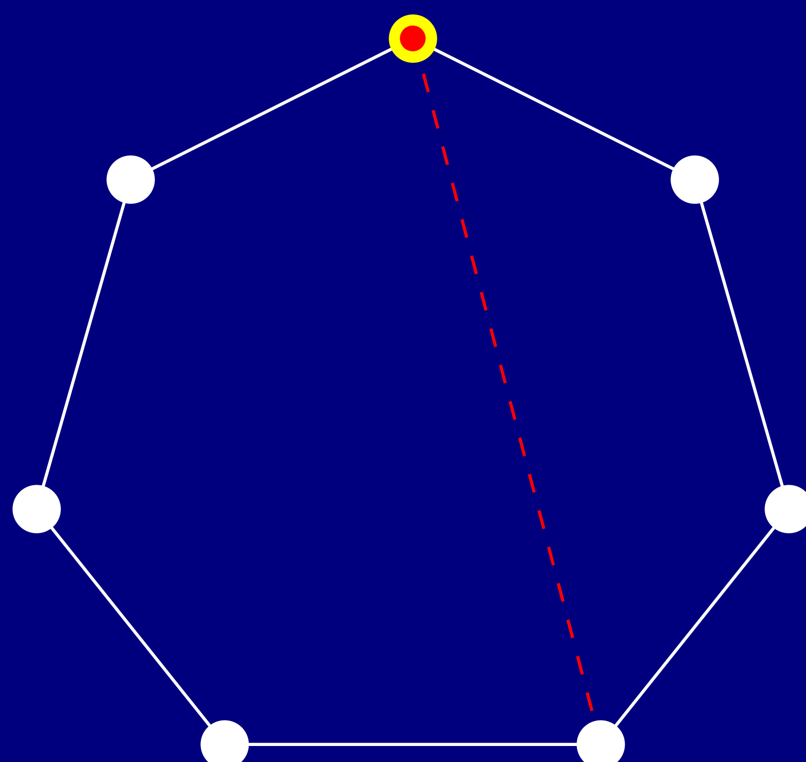

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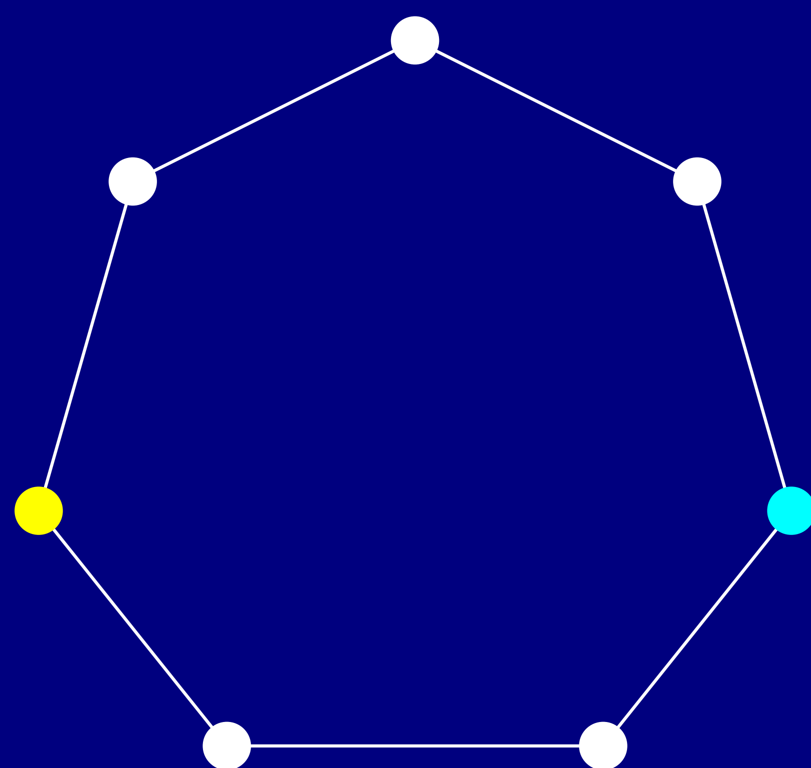

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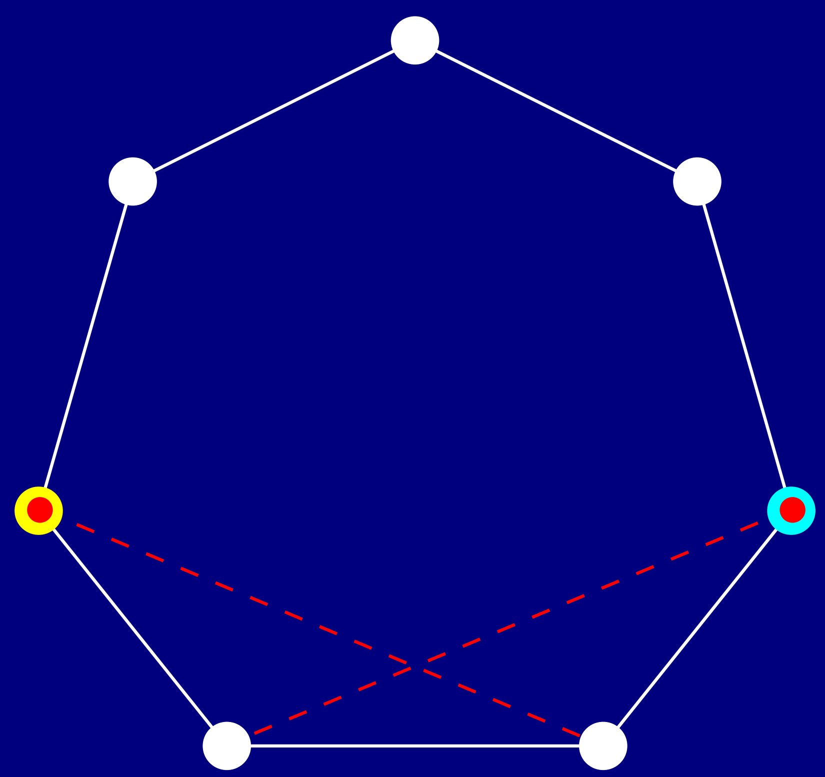

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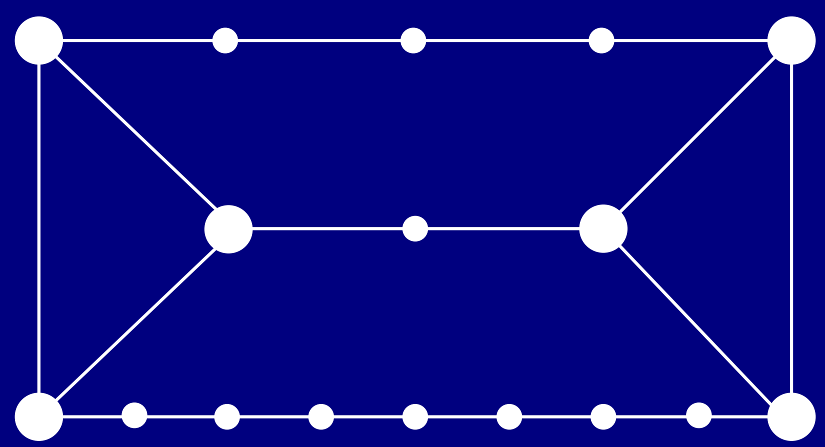

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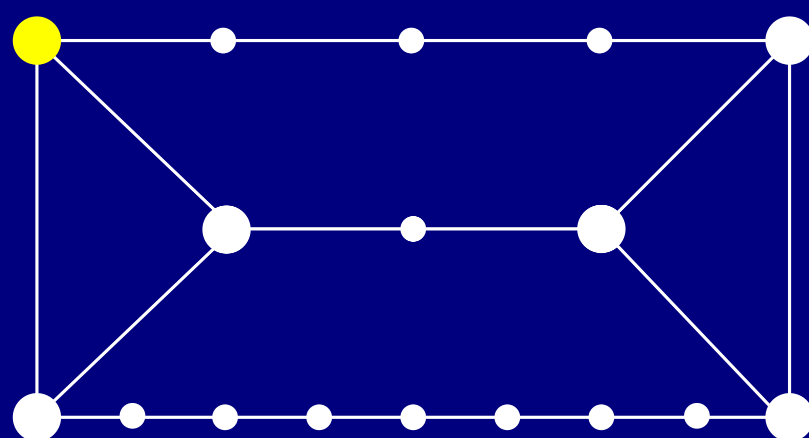

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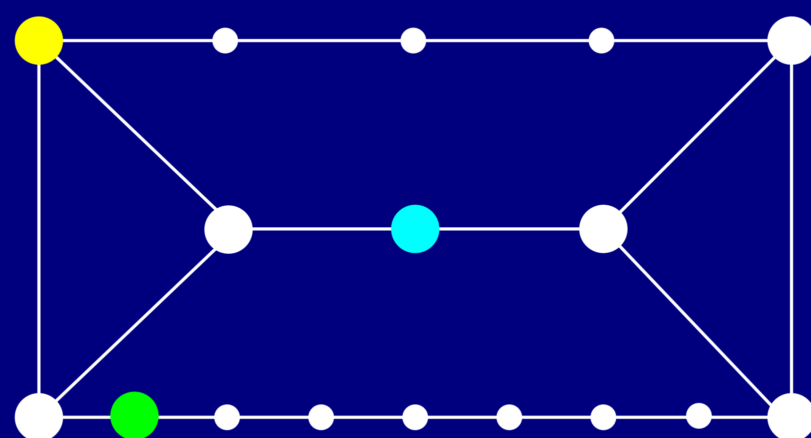

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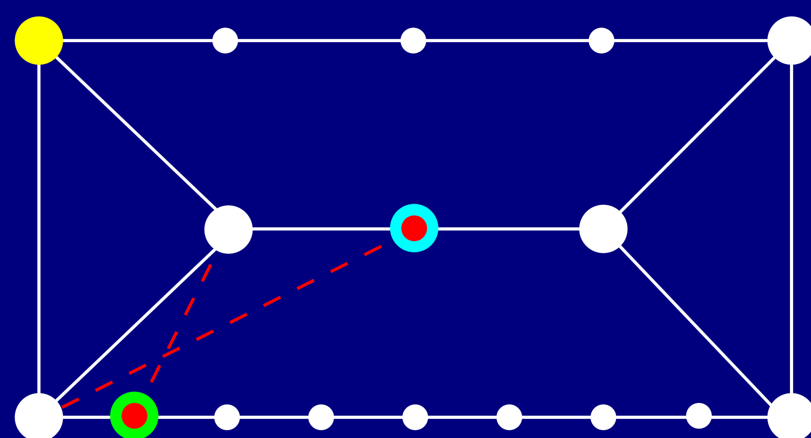

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$\ldots \Longrightarrow \forall \mathcal{Q},\ G^{\mathcal{Q}}$ *is Artemis*

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$\boxed{\text{Combinatorial algorithm for clique-partitioning Artemis graphs?}}$

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Combinatorial algorithm for clique-partitioning Artemis graphs?

Which graphs are actually obtained by co-contracting Meyniel graphs?