On color-critical $(P_5, \overline{P_5})$-free graphs (My collaboration with Frédéric Maffray)

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A synopsis of Fred's work

Fred wrote 125 papers.
Below is a biased synopsis of his work

Fred's first paper

ELSEVIER

Discrete Mathematics

Volume 61, Issues 2–3, September 1986, Pages 247-251

On kernels in *i*-triangulated graphs

F Maffray

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Abstract

A directed graph is said to be kernel-perfect if every induced subgraph possesses a kernel (independent, absorbing subset). A necessary condition for a graph to be kernel-perfect is that every complete subgraph C has an absorbing vertex (i.e., a successor of all vertices of C). In this work, we show that this condition is sufficient for *i*-triangulated graphs, where every odd cycle has two non-crossing chords.

This result appears as a special case of a general relationship between the notion of kernel-perfectness and the well known strong perfect graph conjecture of Berge.

Fred's PhD work

ELSEVIER

Discrete Applied Mathematics
Volume 27, Issues 1–2, May 1990, Pages 85-99

Completely separable graphs ☆

Peter L. Hammer, Frédéric Maffray **

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Abstract

We define a property of Boolean functions called separability, and specialize it for a class of functions naturally associated with graphs. "Completely separable graphs" are then derived and characterized in particular by the existence of two crossing chords in any cycle of length at least five. This implies that completely separable graphs are perfect. We present linear time algorithms for the recognition and for the usual optimization problems (maximum weighted stable set and maximum weighted clique).

Even pairs in weakly chordal graphs

Graphs and Combinatorics
December 1989, Volume 5, Issue 1, pp 339–349 | Cite as

Optimizing weakly triangulated graphs

Authors   Authors and affiliations

Ryan Hayward, Chính Hoàng, Frédéric Maffray

Abstract

A graph is weakly triangulated if neither the graph nor its complement contains a chordless cycle with five or more vertices as an induced subgraph. We use a new characterization of weakly triangulated graphs to solve certain optimization problems for these graphs. Specifically, an algorithm which runs in $O((n + e)n^3)$ time is presented which solves the maximum clique and minimum colouring problems for weakly triangulated graphs; performing the algorithm on the complement gives a solution to the maximum stable set and minimum clique covering problems. Also, an $O((n + e)n^4)$ time algorithm is presented which solves the weighted versions of these problems.

b-perfect graphs

Journal of Graph Theory

Original Article

A Characterization of b-Perfect Graphs

Chính T. Hoàng, Frédéric Maffray, Meriem Mechebbek

First published: 01 December 2011 | https://doi.org/10.1002/jgt.20635 | Cited by: 14

Abstract

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes, and the b-chromatic number of a graph G is the largest integer k such that G admits a b-coloring with k colors. A graph is b-perfect if the b-chromatic number is equal to the chromatic number for every induced subgraph of G. We prove that a graph is b-perfect if and only if it does not contain as an induced subgraph a member of a certain list of 22 graphs. This entails the existence of a polynomial-time recognition algorithm and of a polynomial-time algorithm for coloring exactly the vertices of every b-perfect graph. © 2011 Wiley Periodicals, Inc. J Graph Theory 71:95–122, 2012

Characterization of b-perfect graphs

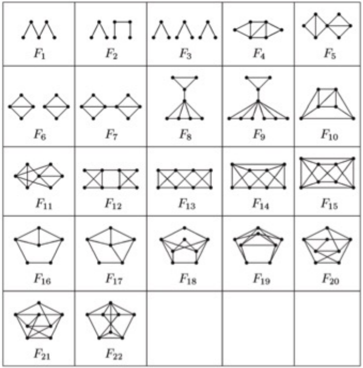

FIGURE 1. Class $\mathcal{F} = \{F_1, \ldots, F_{22}\}$.

Complexity of k-coloring triangle-free graphs

ELSEVIER

Discrete Mathematics
Volume 162, Issues 1–3, 25 December 1996, Pages 313-317

Note

On the NP-completeness of the *k*-colorability problem for triangle-free graphs

Frédéric Maffray [a], Myriam Preissmann [b]

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https://doi.org/10.1016/S0012-365X(97)89267-9

Abstract

We show that the question "Is a graph 3-colorable?" remains NP-complete when restricted to the class of triangle-free graphs with maximum degree 4. Likewise the question "Is a triangle-free graph k-colorable?" is shown to be NP-complete for any fixed value of $k \geqslant 4$.

Perfectly Contractile Graphs

ELSEVIER

Journal of Combinatorial Theory, Series B

Volume 96, Issue 1, January 2006, Pages 1-19

A class of perfectly contractile graphs

Frédéric Maffray [a], Nicolas Trotignon [b, 1]

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Abstract

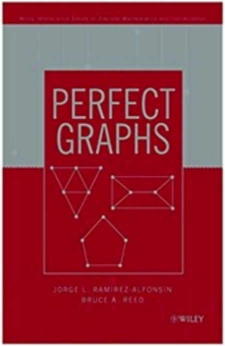
We consider the class $\mathscr{A}$ of graphs that contain no odd hole, no antihole, and no "prism" (a graph consisting of two disjoint triangles with three disjoint paths between them). We prove that every graph $G \in \mathscr{A}$ different from a clique has an "even pair" (two vertices that are not joined by a chordless path of odd length), as conjectured by Everett and Reed ["Even pairs", in: J.L. Ramírez-Alfonsín, B.A. Reed

Translation of Gallai's paper

F. Maffray, M. Preissmann

A translation of Tibor Gallai's article 'Transitiv orientierbare Graphen'

J.L. Ramírez-Alfonsín, B.A. Reed (Eds.), Perfect Graphs, Wiley Interscience, New York (2001)



From 1989 to 2018

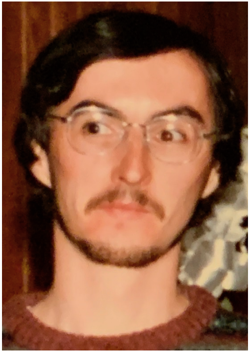


Figure: 1987

Figure: 2015

My collaboration with Fred

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My collaboration with Fred

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H-Free Graphs

Definition

For a graph H, a graph G is H-free, when G **does not contain** H as an **induced** subgraph.

Definition

For a set $\mathcal{L}$ of graphs, a graph G is $\mathcal{L}$-free, when G **does not contain** any graph of $\mathcal{L}$ as an **induced** subgraph.

Complexity of coloring restricted graphs

Theorem (Kral, Kratochvil, Tuza, Woeginger)

Coloring H-Free graphs is

- *polynomial-time solvable if H is an induced subgraph of P_4 or of co-paw, and*
- *NP-hard for any other H*

co-paw = $P_3 \cup K_1$

Theorem (Kral, Kratochvil, Tuza, Woeginger)

Coloring ($2K_2$, co-diamond, $4K_1$, C_5)-free graphs is NP-hard

This implies the hardness results for

- odd-hole-free graphs, or
- P_5-free graphs

k-Coloring P_5-free graphs

Theorem (many authors)

Finding the chromatic number of a P_5-free graph is NP-hard

Theorem (CTH, Marcin Kaminski, Vadim Lozin, Joe Sawada, X. Shu)

For every fixed k, k-coloring a P_5-free graphs is in $\mathcal{P}$

Question: For every fixed k, is the number of k-critical P_5-free graphs finite?

4-crititical P_5-free graphs

Theorem (Daniel Bruce, CTH, Joe Sawada)

The number of 4-critical P_5-free graphs is finite. (6 minimal forbidden **partial** *subgraphs)*

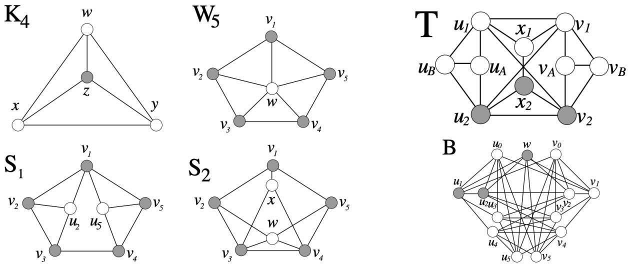


Fig. 1. All 6 MN3P5s

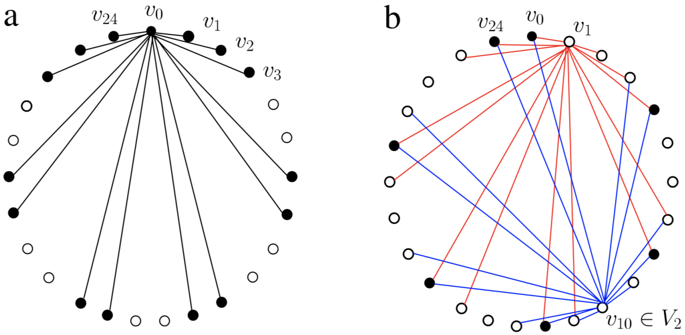
Rediscovered by F. Maffray, G. Morel in 2012

5-critical P_5-free graphs

Theorem (CTH, Brian Moore, Daniel Recoskie, Joe Sawada, Martin Vatshelle 2015)

The number of k-critical ($k \geq 5$) P_5-free graphs is infinite.

Define G_p on 4p +1 vertices, G_p is 5-critical (implying 6-crit, etc)
In the figure below, p = 6



(P_5, co-P_5)-free graphs

Theorem (CTH, D. Adam Lazzarato 2015)

There is an $O(n^3)$ algorithm to find a minimum weighted coloring of a (P_5,co-P_5)-free graph.

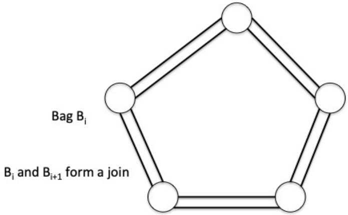
Question: what about k-critical (P_5,co-P_5)-free graphs?

How was the proof found?

Theorem (Harjinder S. Dhaliwal, Angele M. Hamel, CTH, Fredreric Maffray, Tyler J.D. McConnell, Stefan A. Panait)

For every fixed k, the number of k-critical (P_5, co-P_5)-free graphs is finite.

Consider a k-critical (P_5, co-P_5)-free graphs.
If G is perfect then G is the clique K_k.
So G contains a C_5.
Extend the C_5 as much as possible into a buoy



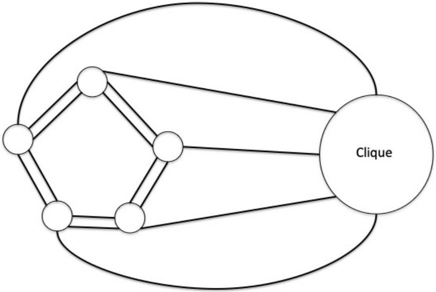
A buoy

Theorem

Let G be a k-critical (P_5, co-P_5)-free graph. Then,

(i) G is a full buoy, or

(ii) G is the join of a buoy and a clique



Join of buoy and clique

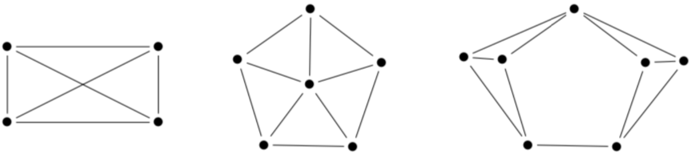
Construction of k-critical (P_5, co-P_5)-free graphs

Let $\mathcal{C}_k$ be the class of k-critical (P_5, co-P_5)-free graphs.

Theorem

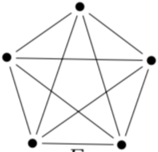
A graph G in $\mathcal{C}_k$ can be constructed from graphs in $\mathcal{C}_{k'}$ by
- *(i) Join of two graphs G_1, $G_2 \in \mathcal{C}_{k'}, k' < k$*
- *(ii) A buoy with bags $B_1, \ldots B_5$ with each $B_i \in \mathcal{C}_{k'}, k' < k$*

4-critical (P_5, co-P_5)-free graphs

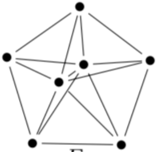


T_1 T_2 T_3

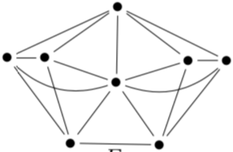
5-critical (P_5, co-P_5)-free graphs



F_1



F_2



F_3

5-critical (P_5, co-P_5)-free graphs

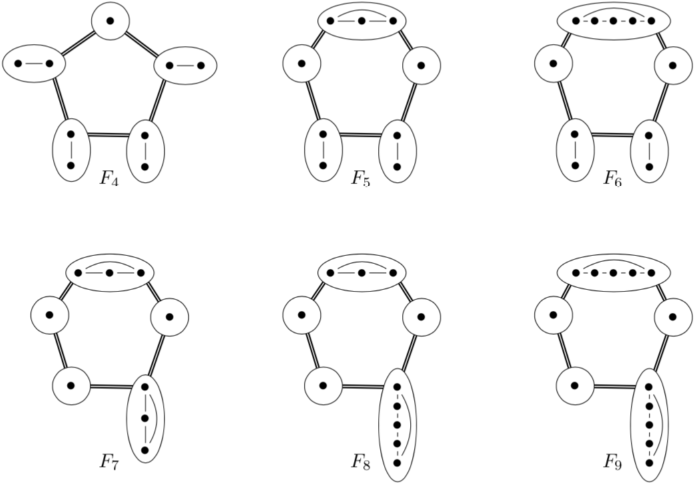
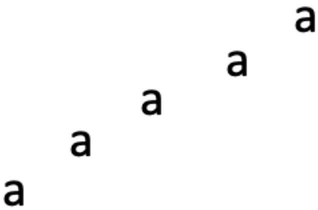


Figure 2: All 5-critical $(P_5, \overline{P}_5)$-free graphs. In the graphs $F_4, \ldots, F_9$, the ovals represent the bags and the double line denotes all edges between the two bags.

Knuth's up arrow notation

$a \uparrow b = a^b$
$a \uparrow\uparrow b = a \uparrow (a \uparrow (\ldots a \uparrow))$

Tower of b a's



The number of k-critical (P_5, co-P_5)-free graphs

Theorem

The number of k-critical (P_5, co-P_5)-free graphs is at most $5 \uparrow\uparrow k$

$$5^{5^{5^{5^{5^{5}}}}}$$

k-critical (P_6, co-P_6)-free graphs

Theorem (Shenwei Huang)

For any fixed $k \geq 5$, k-colorability of P_6-free graphs is NP-hard

Actually, Huang's construction has no co-P_6.

Theorem (Shenwei Huang)

For any fixed $k \geq 5$, k-colorabiity of (P_6, co-P_6)-free graphs is NP-hard

Theorem (Maria Chudnovsky, Sophie Spirkl, Mingxian Zhong)

4-colorability of P_6-free graphs is in $\mathcal{P}$

Questions:
(i) Is the number of 5-critical (P_6, co-P_6)-free graphs finite?
(ii) Characterize 5-critical (P_6, co-P_6)-free graphs

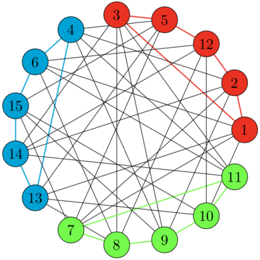
k-critical (P_6, co-P_6)-free graphs

Is the number of 5-critical (P_6, co-P_6)-free graphs finite? Diamond-free graphs are co-P_6-free Is the number of 5-critical (P_6, diamond)-free graphs finite?

Theorem (Kathie Cameraon, CTH, Shenwei Huang and Pablo Morales)

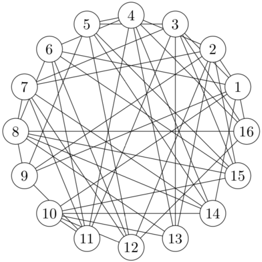
There are 5-crtitical (P_6, diamond)-free graphs with 15, 16, 17 vertices.
There are no 5-crtitical (P_6, diamond)-free graphs with 18 vertices.

5-critical (P_6, diamond)-free graphs



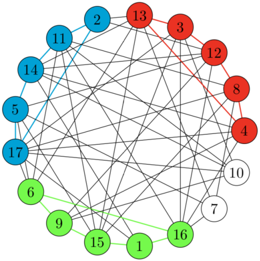
15 vertices

5-critical (P_6, diamond)-free graphs



16 vertices

5-critical (P_6, diamond)-free graphs



17 vertices

Thank you for your attention

Aussois 2015