Keep coloring, even it's hard

A Tribute to Frédéric Maffray

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Computer Science Department
Federal University of Ceará

Grenoble, September 2019

Frédéric Maffray – 19/08/1960-22/08/2018

Figura: Vercors, end of July 2018

Humble Brazilian Tributes to Frédéric

- Seminar at Federal University of Rio de Janeiro, November 2018, organized by Celina de Figueiredo, with talks of Ana Silva and mine.
- Special Session at LAGOS 2019, Belo Horizonte, June 2019, with talks of Chinh Hoáng and Celina de Figueiredo.
- Two Special Sessions at the First Brazilian-French Workshop at the Institute of Pure and Applied Mathematics, Rio de Janeiro, July 2019, organized by us.

The vertex coloring problem

k-coloring

Given a graph $G = (V, E)$, a (vertex) k-coloring of G is an assignment $c : V(G) \longmapsto \{1, \cdots, k\}$ in such a way that $c(u) \neq c(v)$, whenever $uv \in E(G)$.

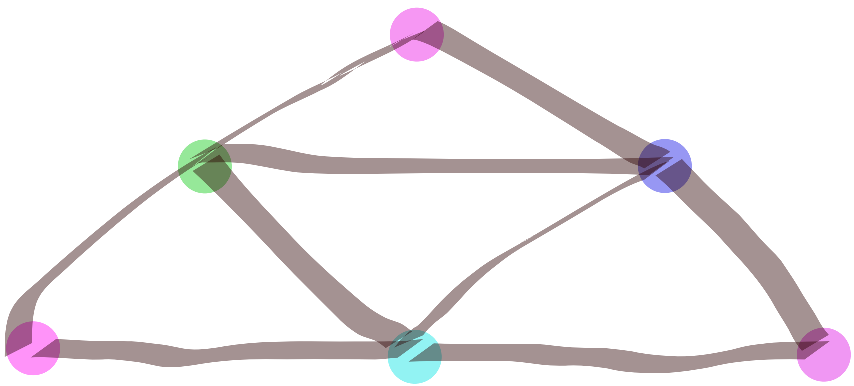

Figura: 4-coloring of the Hajös graph

The vertex coloring problem

Chromatic number

Given a graph $G = (V, E)$, the chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G admits a k-coloring.

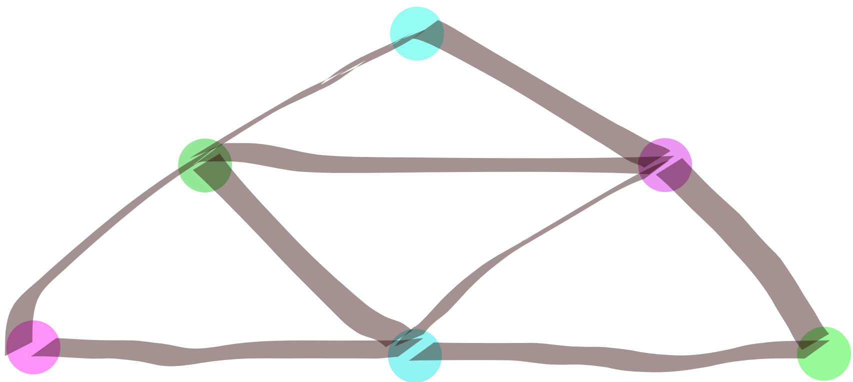

Figura: 3-coloring of the Hajös graph

The vertex coloring problem

- The vertex coloring problem and several of its applications are used to model a significant number of practical problems related to partition sets according to conflicts or similarities, such as scheduling, frequency assignment. etc. And, it's nice.

- We don't know a polynomial algorithm to determine $\chi(G)$ for a general graph G. Actually, the problem is *NP*-hard, even for triangle-free graphs, and it's hard to approximate. And so, it's challenging.

F. Maffray, M. Preissmann. On the NP-completeness of the k-colorability problem for triangle-free graphs. Discrete Mathematics 162 (1996), 313-317.

C. Lund and M. Yannakakis. On the hardness of approximating minimization problems. Journal of the ACM, 41(1994) (5):960–981.

Given a graph $G = (V, E)$, we denote by $\omega(G)$ the size of the largest clique of G. A **hole** is a chordless cycle of length at least 5. An **antihole** is the complement of a hole.

Perfect Graphs

A graph G is perfect if for every induced subgraph H of G, $\chi(H) = \omega(H)$

C. Berge, Some classes of perfect graphs, in: Six Papers on Graph Theory (Indian Statistical Institute, Calcutá, 1963), 1-21.

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Strong Perfect Graph Conjecture

A graph G is perfect if and only if G does not contain an odd hole nor an odd antihole as induced subgraph.

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Strong Perfect Graph Conjecture

A graph G is perfect if and only if G does not contain an odd hole nor an odd antihole as induced subgraph.

Berge Graphs

A graph G is Berge if and only if G does not have an odd hole nor its complement as induced subgraph.

C. Berge, Some classes of perfect graphs, in: Six Papers on Graph Theory (Indian Statistical Institute, Calcutá, 1963), 1-21.

Theorem (Perfect Graph Theorem)

A graph G is perfect if and only if $\bar{G}$ is perfect.

Theorem (Strong Perfect Graph Theorem)

A graph G is perfect if and only if it is Berge.

L. Lovász. Normal hypergraphs and the perfect graph conjecture, Discrete Mathematics, 2 (1972) (3): 253–267.

M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas. The strong perfect graph theorem, Annals of Mathematics, 164 (2006) (1): 51–229.

- If G is perfect, $\chi(G)$ can be determined in polynomial time.

- The algorithm that determines $\chi(G)$ does not have a reputation to be useful in practice. And, we can say that it belongs to the "combinatorial optimization world".

Open problem:

Finding a polynomial time algorithm from the "combinatorial graph theory world" to determine $\chi(G)$, G perfect. And a good one in practice.

Still an open problem!

M. Grötschel, L. Lovász and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981), 169- 197.

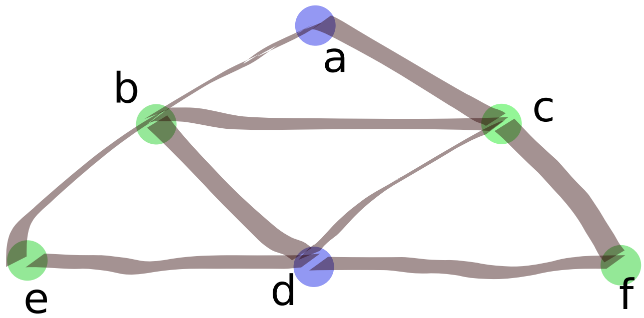
Even pairs or Paires d'amis

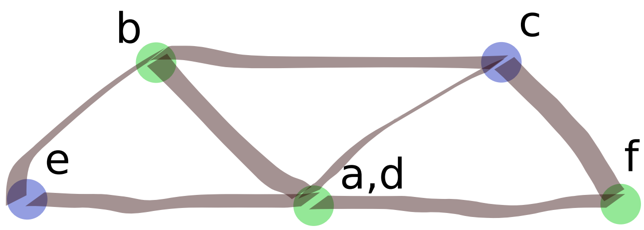
- An even pair (paire d'amis) in a graph $G = (V, E)$ is a pair of non-adjacent vertices of G such that every induced path between them has an even number of edges.
- Fonlupt and Uhry proved that contracting an even pair in a perfect graph G preserves perfection and the size of the maximum clique of G, therefore its chromatic number.
- Meyniel proved that no minimal imperfect graph has an even pair.

J. Fonlupt, J.P. Uhry, Transformations which preserve perfectness and H-perfectness of graphs. Annals of Discr. Math., 16 (1982), pp. 83–95.

H. Meyniel, A new property of critical imperfect graphs and some consequences. European J. Comb. 8 (1987) 313-316.

A vertex coloring algorithm





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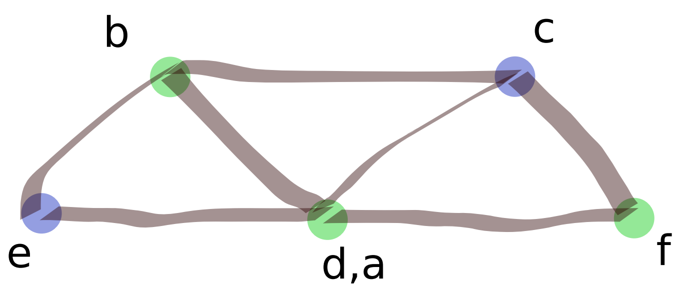

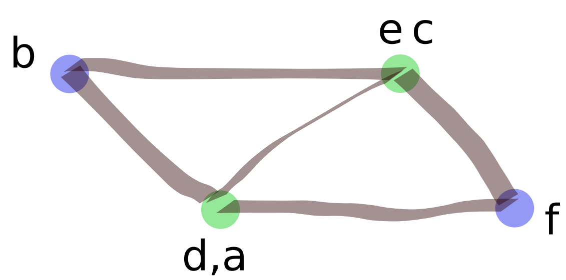

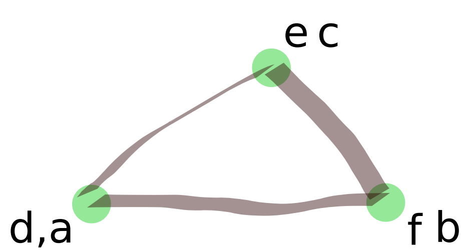

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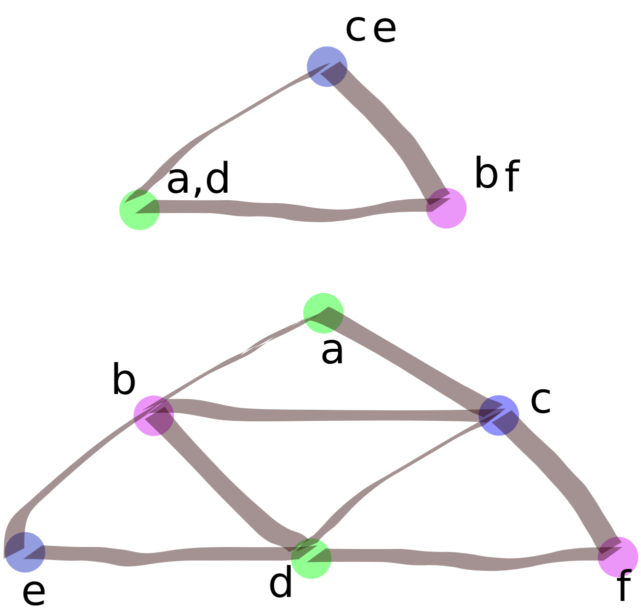
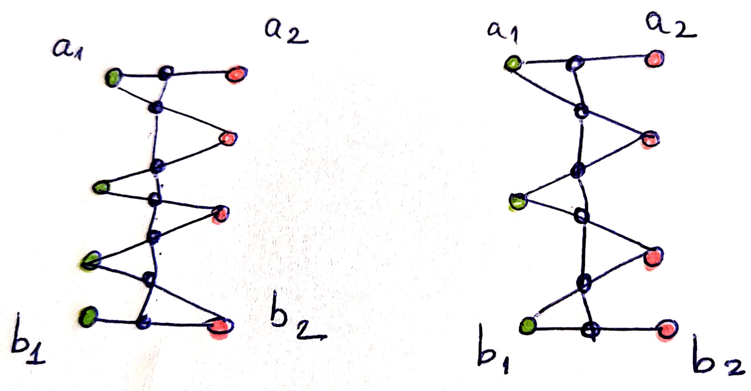


Classes of Perfect Friendly Graphs

A graph G is said to be of strict quasi-parity (SQP) if G and each induced subgraph of G either is complete or has an even pair.

Theorem (Hourgardy,1991)

If H is 3-connected bipartite graph, then the line graph of H has no even pairs.



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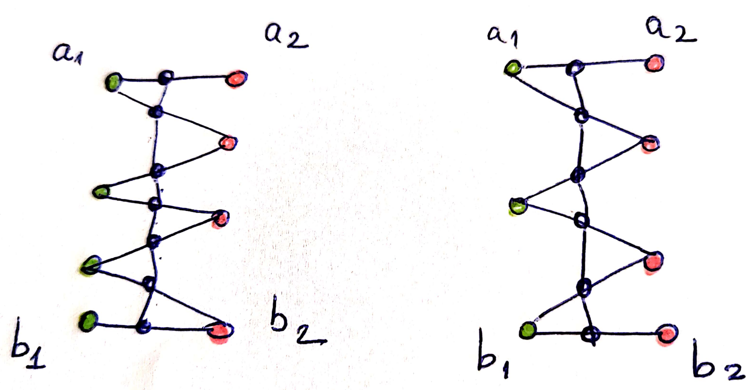

Conjecture of S. Hougardy

If G is a minimal non-SQP graph, then G is an odd hole, or an antihole, or a line graph of a bipartite graph.

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Bertshi defined a graph G be perfectly contractile (PC) if G and each induced subgraph of G has a sequence of even pair contractions leading it to a complete graph.

Conjecture of H. Everett and B. Reed

A graph G is a PC graph if and only if G has no odd hole, no antihole and no odd prism.

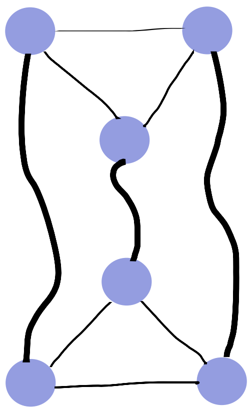

Figura: Prism

M. Bertshi. Perfectly Contractile Graphs. Journal of Combinatorial Theory Series B 50 (1990) 222-230.

Reed B.A., Problem session on parity problems (Public communication). DIMACS Workshop on Perfect Graphs, Princeton University, New Jersey, 1993.

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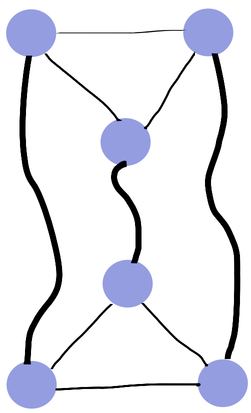

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Hougardy and Everett&Reed' conjectures are still open!

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Planar graphs

Theorem (LS, Maffray, Reed, 2008)

Every minimal planar non-SQP graph is an odd hole or a line graph of a bipartite graph.

We actually characterized the family of minimal non-SQP which are line graph of bipartite graphs.

Linhares Sales, C.; Maffray, F. ; Reed, B. A., On Planar Strict Quasi Parity Graphs. SIAM Journal on Discrete Mathematics, v. 22, p. 329-347, 2008.

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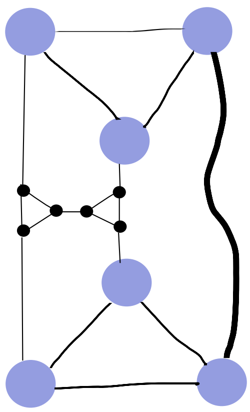

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We've used Hsu's decomposition of planar perfect graphs to recognize planar PC graphs.

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Linhares Sales, C.; Maffray, F. ; Reed, B. A., On Planar Perfectly Contractile Graphs. Graphs and Combinatorics,, v. 13, p. 167-187, 1997.

Hsu, W.L.: Recognizing planar perfect graphs. J. Assoc. Compo Mach. 34, 255-288 (1987).

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After decomposing a planar graph G by clique cutsets, cutsets of size two, three and four, the non-complete basic graphs are examined. If G is perfect, the basic graphs are line graphs of bipartite graphs or quasi line graphs of bipartite graphs or comparability graphs.

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Claw-free graphs

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Every minimal non-SQP claw-free graph is either an odd hole or antihole or a line graph of a bipartite graph.

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V. Chvátal and N. Sbihi, Recognizing claw-free perfect graphs, J. Combin. Theory, Ser. B 44 (1988), 154-176.

Linhares Sales, C.; Maffray, F., Even Pairs In Claw-Free Perfect Graphs. Journal of Combinatorial Theory. Series B, UNITED STATES, v. 74, p. 169-191, 1998.

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Theorem (Chvátal, Sbihi, 1988)

*Let $G = (V, E)$ be a claw-free graph without any clique cutest. Then G is perfect if and only if G is **peculiar or elementary**.*

V. Chvátal and N. Sbihi, Recognizing claw-free perfect graphs, J. Combin. Theory, Ser. B 44 (1988), 154-176.

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Claw-free graphs

Elementary and peculiar graphs

A graph $G = (V, E)$ is elementary if its edges can be colored with 2 colors in such a way that every P_3 is bicolored. A graph G is peculiar if $V(G)$ can be partitioned in 6 sets with very well defined adjacencies between them.

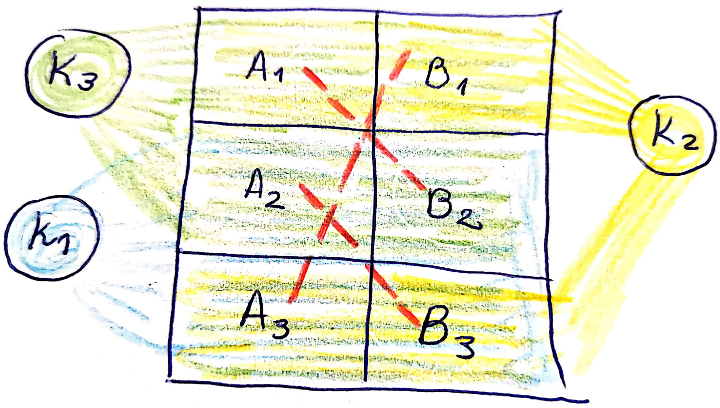


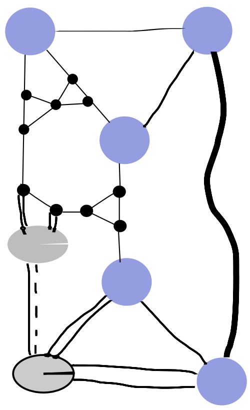
Figura: Peculiar Graphs

Peculiar graphs without antiholes are perfectly orderable graphs and then perfectly contractile graphs.

And then, we've used the characterization of elementary graphs of Bruce and Frédéric

Theorem (Maffray, Reed, 1999)

A graph G is elementary if and only if it's an augmentation of line graph of a multibipartite graph.



F. Maffray, B.A. Reed. A description of claw-free perfect graphs. Journal of Combinatorial Theory B, 75 (1999), 134-156.

C_4-free graphs

Theorem (LS, Maffray, 2004)

Let G be a C_4-free Berge graph without any prisms. Then G is either complete or has an even pair.

Linhares Sales C.; Maffray, F. Even Pairs in Square-free Berge Graphs. Matemática Contemporânea 25 (2003) 161-176.

F. Maffray, N. Trotignon. A class of perfectly contractile graphs. Journal of Combinatorial Theory, Series B, Volume 96 (2006), 1-19.

M. Chudnovsky, I. Lob, F. Maffray, N. Trotignon, K. Vuskovic. Coloring square-free Berge graphs. Journal of Combinatorial Theory, Series B, Volume 135 (2019), 96–128.

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Theorem (Chudnovsky, Lob, Maffray, Trotignon, Vuskovic, 2019)

There is a polynomial algorithm from the "combinatorial combinatorial world" to color C4-free perfect graphs.

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F. Maffray, N. Trotignon. A class of perfectly contractile graphs. Journal of Combinatorial Theory, Series B, Volume 96 (2006), 1-19.

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More *paires d'amis*

Theorem (LS, Maffray, 2004)

A dart-free graph is PC if and only if has no odd holes, no antiholes and no odd prisms.

Theorem (de Figueiredo, Maffray, Villela, 2006)

Let $G = (V, E)$ be a perfect bull-reducible graph with at least two vertices. Then G or $\bar{G}$ has an even pair, i.e., G is QP.

C.M.H. de Figueiredo, F. Maffray, C.R. Villela Maciel. Even pairs in bull-reducible graphs. In: Graph Theory in Paris, Proc. Conf. in Memory of Claude Berge. A. Bondy, J. Fonlupt, J.-L. Fouquet, J.-C. Fournier, J.L. Ramírez-Alfonsín (Eds.), Trends in Mathematics, Birkhäuser Verlag, 2006, pp. 179-195.

Linhares Sales C.; Maffray, F. . On Dart-free Perfectly Contractile Graphs. Theoretical Computer Science, Amsterdan, v. 321, p. 171-194, 2004.

Random Sample of Paires d'Amis

Figura: Marielle, Bruno, Frédéric and Hugo — Grenoble, February 2007

A bunch of *paires d'amis*

More *paires d'amis*

Figura: Celina+Fred = 100, 2010

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Unexpected results of paires d'amis

Research group ParGO at Federal University of Ceará - Brazil

Figura: ParGO 20 - Opening session

Figura: ParGO 20 years - The audience