Unavoidability of trees in tournaments

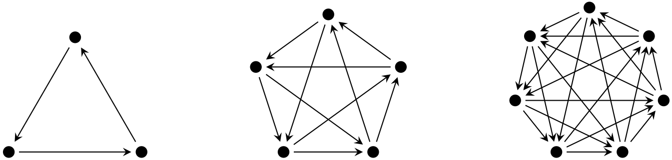
François Dross and Frédéric Havet

COATI, I3S, CNRS, INRIA, Univ. Côte d'Azur
Sophia Antipolis, France

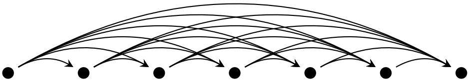
A tribute to Frédéric Maffray, September 02-04 2019

Tournament

tournament $=$ orientation of a complete graph.



transitive tournament $=$ tournament with no directed cycle
TT_n $=$ transitive tournament of order n.

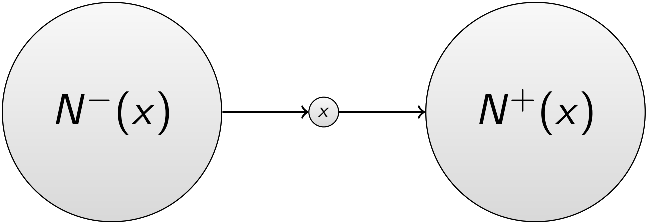


Redei's Theorem

Theorem (Redei, 1934) Every tournament has a directed Hamiltonian path.

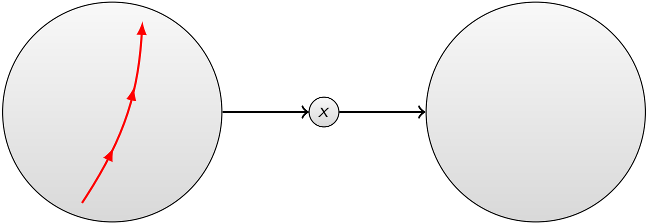
Redei's Theorem

Theorem (Redei, 1934) Every tournament has a directed Hamiltonian path.



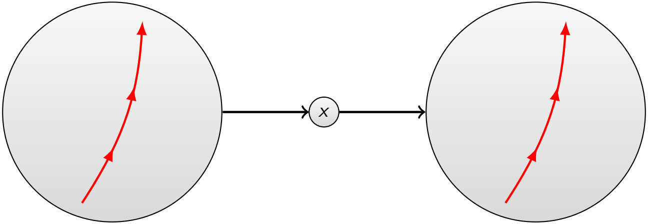
Redei's Theorem

Theorem (Redei, 1934) Every tournament has a directed Hamiltonian path.



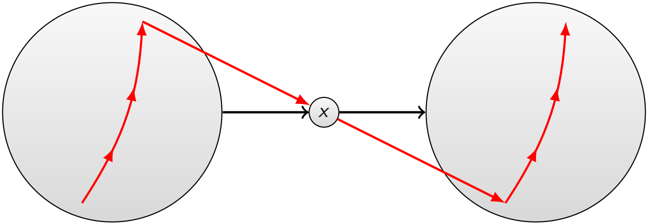
Redei's Theorem

Theorem (Redei, 1934) Every tournament has a directed Hamiltonian path.



Redei's Theorem

Theorem (Redei, 1934) Every tournament has a directed Hamiltonian path.



Unavoidability

n-unavoidable $=$ contained in every tournament of order n

unavoidable $=$ n-unavoidable for some n.

$\mathrm{unvd}(D)$: unavoidability $=$ minimum n s.t. D is n-unavoidable.

Redei's Theorem: $\mathrm{unvd}(\vec{P}_n) = n$. $\vec{P}_n$: directed path of order n.

Q1: Which digraphs are unavoidable ?

Q2: For an unavoidable digraph D, what is $\mathrm{unvd}(D)$?

Unavoidable digraphs

D is unavoidable if and only if D is acyclic.

- ▶ unavoidable $\Rightarrow$ contained in some TT_p $\Rightarrow$ no directed cycle
- ◀ every acyclic digraph of order n is contained in TT_n.
  Suffices to prove it for transitive tournaments.

$$\mathrm{unvd}(TT_n) \leq 2\,\mathrm{unvd}(TT_{n-1})$$

[[*Proof* : A tournament of order $2\,\mathrm{unvd}(TT_{n-1})$ contains a vertex with $d^+ \geq \mathrm{unvd}(TT_{n-1})$.]]

Corollary $\mathrm{unvd}(TT_n) \leq 2^{n-1}$.

Upper bounds on unvd(TT_n)

$$\text{unvd}(TT_n) \leq 2^{n-1}.$$

unvd(TT_1) $= 1$, unvd(TT_2) $= 2$, unvd(TT_3) $= 4$, and
unvd(TT_4) $= 8$ (because of Paley tournament).
Reid and Parker, 1970 : unvd(TT_5) $= 14$, unvd(TT_6) $= 28$.
Sanchez-Flores, 1994 : unvd(TT_7) $= 54$.

Corollary unvd(TT_n) $\leq 54 \times 2^{n-7}$ (for $n \geq 7$).

Lower bounds on unvd(TT_n)

Theorem (Erdős and Moser, 1964) unvd$(TT_n) > 2^{(n-1)/2}$.

[[*Proof* : Random tournament T on $p = 2^{(n-1)/2}$ vertices.
Probability that $T\langle v_1, \dots, v_n\rangle$ is transitive with hamiltonian dipath $(v_1, \dots, v_n)$ is $\left(\frac{1}{2}\right)^{\binom{n}{2}}$.
Expected number of transitive n-tournaments : $\frac{p!}{(p-n)!}\left(\frac{1}{2}\right)^{\binom{n}{2}}$

$$< p^n \left(\frac{1}{2}\right)^{\binom{n}{2}} \leq 1.$$

First Moment Method, p-tournament with no TT_n.]]

Theorem For every $C > 1$, $C \times$ unvd$(TT_n) > 2^{(n+1)/2}$ if n is large enough.

[[Use Local Lemma]]

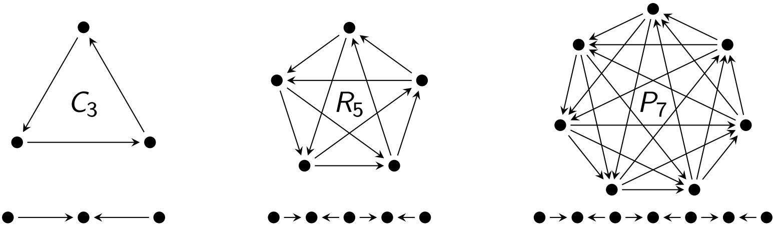
Value of unvd(TT_n)

Question : What is the value of unvd(TT_n) ?

$$2^{(n-1)/2} < \text{unvd}(TT_n) \leq 2^{n-1}$$

Unavoidability of oriented paths

$\vec{P}_n$: directed path on n vertices.
Theorem (Redei, 1934) $\text{unvd}(\vec{P}_n) = n$.



Theorem (H. and Thomassé, 2000) $\text{unvd}(P) = |P|$ if $|P| \geq 8$.
T tournament, P oriented path with $|T| = |P|$.
T contains P unless $T \in \{C_3, R_5, P_7\}$ and P is antidirected.

Unavoidablity of cycles

Recall directed cycles are non-unavoidable, oriented cycles are non-universal.

Theorem (Thomason, 1986)
If C is a non-directed cycle with $|C| \geq \mathbf{2^{128}}$, then $\mathrm{unvd}(C) = |C|$.

Theorem (H. , 2000)
If C is an non-directed cycle with $|C| \geq \mathbf{68}$, then $\mathrm{unvd}(C) = |C|$.

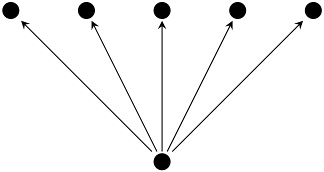
Conjecture
If C is an non-directed cycle with $|C| \geq \mathbf{9}$, then $\mathrm{unvd}(C) = |C|$.

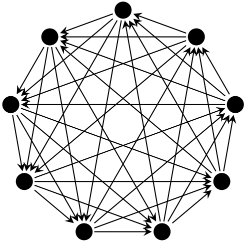
Unavoidability of oriented trees

Conjecture (Sumner, 1972)
Every oriented tree of order n is $(2n-2)$-unavoidable.

Unavoidability of oriented trees

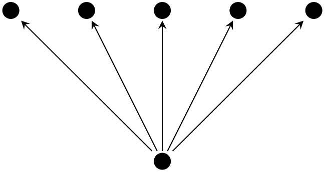
Conjecture (Sumner, 1972)
Every oriented tree of order n is $(2n-2)$-unavoidable.

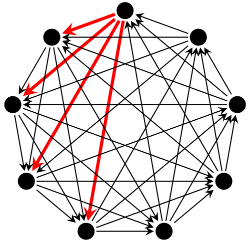




Unavoidability of oriented trees

Conjecture (Sumner, 1972)
Every oriented tree of order n is $(2n-2)$-unavoidable.





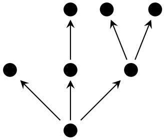
Unavoidability of oriented trees

Conjecture (Sumner, 1972).
If T is an oriented tree of order n, then $\mathrm{unvd}(T) \leq 2n-2$.

If T is an oriented tree of order n, then $\mathrm{unvd}(T) \leq$

(Häggkvist and Thomason, 1991)	$12n$	$(4+o(1))n$
(H. and Thomassé, 2000)	$\frac{7}{2}n - \frac{5}{2}$	
(El Sahili, 2004)	$3n-3$	
(Kühn, Mycroft and Osthus, 2011)	$2n-2$ for n large	.

Theorem (H. and Thomassé, 2000).
If A is an arborescence, then $\mathrm{unvd}(A) \leq 2|A|-2$.



Beyond Sumner's conjecture

Conjecture (H. and Thomassé, 2000).
If T is an oriented tree of order n with k leaves, then

$$\mathrm{unvd}(T) \leq n + k - 1.$$

Evidences : True for $k \leq 3$. (Ceroi and H., 2004).

True for a large class of trees. (H. 2002) .

$\mathrm{unvd}(T) \leq n + 2^{512k^3}$. (Häggkvist and Thomason, 1991)

Our results

Theorem (Dross and H. , 2018).
If A is an arborescence of order n with k leaves,
then $\text{unvd}(A) \leq n + k - 1.$

Theorem (Dross and H. , 2018).
If T is a tree of order n with k leaves, then

$$\text{unvd}(T) \leq \begin{cases} \frac{3}{2}n + \frac{3}{2}k - 2 \\ \\ \\ \end{cases}$$

Our results

Theorem (Dross and H. , 2018).
If A is an arborescence of order n with k leaves,
then $\text{unvd}(A) \leq n + k - 1.$

Theorem (Dross and H. , 2018).
If T is a tree of order n with k leaves, then

$$\text{unvd}(T) \leq \begin{cases} \frac{3}{2}n + \frac{3}{2}k - 2 & \Rightarrow \text{Sumner holds} \\ & \quad \text{when } k \leq n/3 \end{cases}$$

Our results

Theorem (Dross and H. , 2018).
If A is an arborescence of order n with k leaves,
then $\text{unvd}(A) \leq n + k - 1.$

Theorem (Dross and H. , 2018).
If T is a tree of order n with k leaves, then

$$\text{unvd}(T) \leq \begin{cases} \frac{3}{2}n + \frac{3}{2}k - 2 \\ \\ \frac{9}{2}n - \frac{5}{2}k - \frac{9}{2} \\ \\ \end{cases}$$

Our results

Theorem (Dross and H. , 2018).
If A is an arborescence of order n with k leaves,
then $\mathrm{unvd}(A) \leq n + k - 1.$

Theorem (Dross and H. , 2018).
If T is a tree of order n with k leaves, then

$$\mathrm{unvd}(T) \leq \left\{ \begin{array}{l} \frac{3}{2}n + \frac{3}{2}k - 2 \\ \\ \frac{9}{2}n - \frac{5}{2}k - \frac{9}{2} \end{array} \right\} \Longrightarrow \frac{21}{8}n - \frac{47}{16}$$

Our results

Theorem (Dross and H. , 2018).
If A is an arborescence of order n with k leaves,
then $\mathrm{unvd}(A) \leq n + k - 1.$

Theorem (Dross and H. , 2018).
If T is a tree of order n with k leaves, then

$$\mathrm{unvd}(T) \leq \begin{cases} \left.\begin{array}{l} \frac{3}{2}n + \frac{3}{2}k - 2 \\ \\ \frac{9}{2}n - \frac{5}{2}k - \frac{9}{2} \end{array}\right\} \Longrightarrow \frac{21}{8}n - \frac{47}{16} \\ \\ n + 144k^2 - 280k + 124 \end{cases}$$

Median orders

median order : $(v_1, v_2, \ldots, v_n)$ s.t. $|\{(v_i, v_j) : i < j\}|$ is maximum.

Proposition : If $(v_1, v_2, \ldots, v_n)$ is a median order of T, then

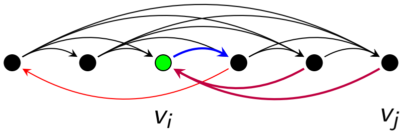
(M) v_i dominates at least half of the vertices $v_{i+1}, \ldots, v_j$, and
v_j is dominated by at least half of the vertices $v_i, \ldots, v_{j-1}$.

Median orders

median order : $(v_1, v_2, \ldots, v_n)$ s.t. $|\{(v_i, v_j) : i < j\}|$ is maximum.

Proposition : If $(v_1, v_2, \ldots, v_n)$ is a median order of T, then

(M) v_i dominates at least half of the vertices $v_{i+1}, \ldots, v_j$, and
v_j is dominated by at least half of the vertices $v_i, \ldots, v_{j-1}$.

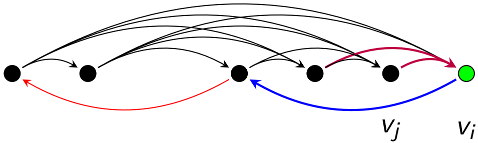


Median orders

median order : $(v_1, v_2, \ldots, v_n)$ s.t. $|\{(v_i, v_j) : i < j\}|$ is maximum.

Proposition : If $(v_1, v_2, \ldots, v_n)$ is a median order of T, then

(M) v_i dominates at least half of the vertices $v_{i+1}, \ldots, v_j$, and
v_j is dominated by at least half of the vertices $v_i, \ldots, v_{j-1}$.



$\mathrm{unvd}(A) \leq n + k - 1$: the greedy procedure

A arborescence with root r, n nodes, k leaves.
$(v_1, \ldots, v_m)$ median order of T with $|T| = m = n + k - 1$.

Set $\phi(r) = v_1$.

For $i = 1$ to m, do

- if v_i is not hit, skip; v_i is failed $(v_i \in F)$
- if v_i is hit, let $a_i = \phi^{-1}(v_i)$;
  assign the $|N^+(a_i)|$ first not yet hit out-neighbours of v_i in $\{v_{i+1}, \ldots, v_m\}$ to the sons of a_i (according to some predefined order);

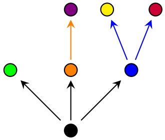
$\mathrm{unvd}(A) \leq n + k - 1$: the greedy procedure

A arborescence with root r, n nodes, k leaves.
$(v_1, \ldots, v_m)$ median order of T with $|T| = m = n + k - 1$.

Set $\phi(r) = v_1$.

For $i = 1$ to m, do

- if v_i is not hit, skip; v_i is failed $(v_i \in F)$
- if v_i is hit, let $a_i = \phi^{-1}(v_i)$;
  assign the $|N^+(a_i)|$ first not yet hit out-neighbours of v_i in $\{v_{i+1}, \ldots, v_m\}$ to the sons of a_i (according to some predefined order);



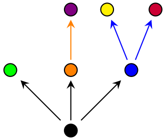
$\mathrm{unvd}(A) \le n+k-1$: the greedy procedure

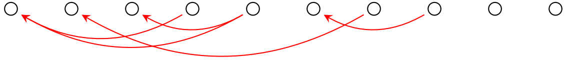
A arborescence with root r, n nodes, k leaves.
$(v_1, \dots, v_m)$ median order of T with $|T| = m = n + k - 1$.

Set $\phi(r) = v_1$.

For $i = 1$ to m, do

- if v_i is not hit, skip; $\quad$ v_i is failed ($v_i \in F$)
- if v_i is hit, let $a_i = \phi^{-1}(v_i)$;
  assign the $|N^+(a_i)|$ first not yet hit out-neighbours of v_i in $\{v_{i+1}, \dots, v_m\}$ to the sons of a_i (according to some predefined order);





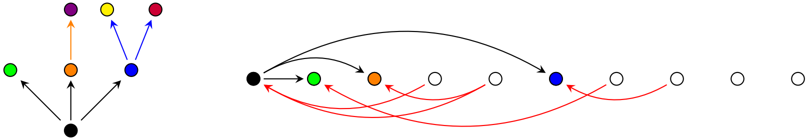
$\mathrm{unvd}(A) \leq n + k - 1$: the greedy procedure

A arborescence with root r, n nodes, k leaves.
$(v_1, \ldots, v_m)$ median order of T with $|T| = m = n + k - 1$.

Set $\phi(r) = v_1$.

For $i = 1$ to m, do

- if v_i is not hit, skip; v_i is failed ($v_i \in F$)
- if v_i is hit, let $a_i = \phi^{-1}(v_i)$;
  assign the $|N^+(a_i)|$ first not yet hit out-neighbours of v_i in $\{v_{i+1}, \ldots, v_m\}$ to the sons of a_i (according to some predefined order);

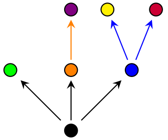


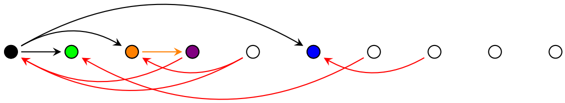
$\mathrm{unvd}(A) \leq n+k-1$: the greedy procedure

A arborescence with root r, n nodes, k leaves.
$(v_1, \ldots, v_m)$ median order of T with $|T| = m = n+k-1$.

Set $\phi(r) = v_1$.
For $i = 1$ to m, do

- if v_i is not hit, skip; $\qquad$ v_i is failed ($v_i \in F$)
- if v_i is hit, let $a_i = \phi^{-1}(v_i)$;
  assign the $|N^+(a_i)|$ first not yet hit out-neighbours of v_i in $\{v_{i+1}, \ldots, v_m\}$ to the sons of a_i (according to some predefined order);





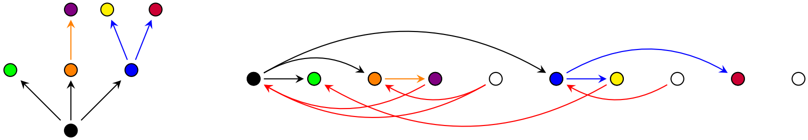
$\mathrm{unvd}(A) \leq n + k - 1$: the greedy procedure

A arborescence with root r, n nodes, k leaves.
$(v_1, \dots, v_m)$ median order of T with $|T| = m = n + k - 1$.

Set $\phi(r) = v_1$.

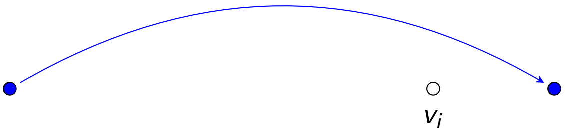
For $i = 1$ to m, do

- if v_i is not hit, skip; v_i is failed ($v_i \in F$)
- if v_i is hit, let $a_i = \phi^{-1}(v_i)$;
  assign the $|N^+(a_i)|$ first not yet hit out-neighbours of v_i in $\{v_{i+1}, \dots, v_m\}$ to the sons of a_i (according to some predefined order);



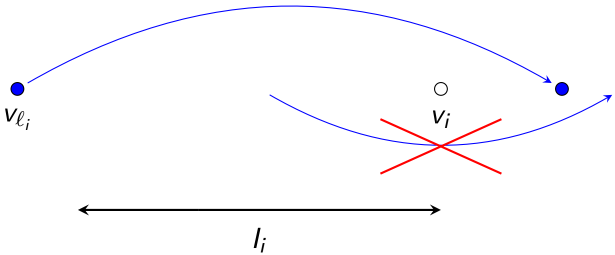
$\mathrm{unvd}(A) \leq n + k - 1$: analysis

vertex v_ℓ is active for i if $v_\ell = \phi(a)$ for node a and a has a son b that is not embedded in $\{v_1, \ldots, v_i\}$.
For $v_i \in F$, let ℓ_i be the largest index such that v_{ℓ_i} is active for i.
Set $I_i = \{v_{\ell_{i+1}}, \ldots, v_i\}$.



$\mathrm{unvd}(A) \leq n + k - 1$: analysis

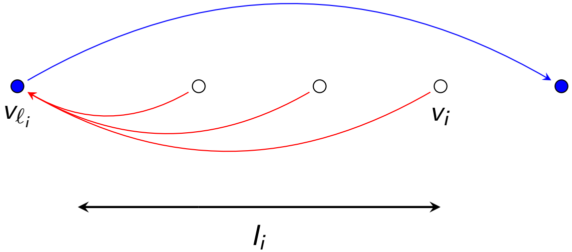
vertex v_ℓ is active for i if $v_\ell = \phi(a)$ for node a and a has a son b that is not embedded in $\{v_1, \ldots, v_i\}$.
For $v_i \in F$, let ℓ_i be the largest index such that v_{ℓ_i} is active for i.
Set $I_i = \{v_{\ell_{i+1}}, \ldots, v_i\}$.



$\mathrm{unvd}(A) \leq n + k - 1$: analysis

vertex v_ℓ is active for i if $v_\ell = \phi(a)$ for node a and a has a son b that is not embedded in $\{v_1, \ldots, v_i\}$.
For $v_i \in F$, let ℓ_i be the largest index such that v_{ℓ_i} is active for i.
Set $I_i = \{v_{\ell_{i+1}}, \ldots, v_i\}$.

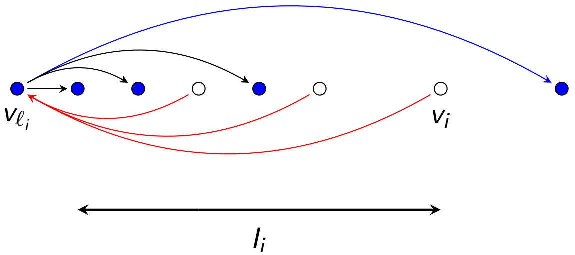
Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $L = \{\text{out-leaves}\}$.



$\mathrm{unvd}(A) \leq n + k - 1$: analysis

vertex v_ℓ is active for i if $v_\ell = \phi(a)$ for node a and a has a son b that is not embedded in $\{v_1, \ldots, v_i\}$.
For $v_i \in F$, let ℓ_i be the largest index such that v_{ℓ_i} is active for i.
Set $I_i = \{v_{\ell_{i+1}}, \ldots, v_i\}$.

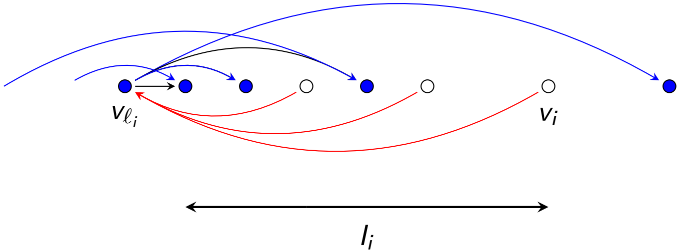
Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $L = \{\text{out-leaves}\}$.



$\mathrm{unvd}(A) \leq n + k - 1$: analysis

vertex v_ℓ is active for i if $v_\ell = \phi(a)$ for node a and a has a son b that is not embedded in $\{v_1, \ldots, v_i\}$.
For $v_i \in F$, let ℓ_i be the largest index such that v_{ℓ_i} is active for i.
Set $I_i = \{v_{\ell_{i+1}}, \ldots, v_i\}$.

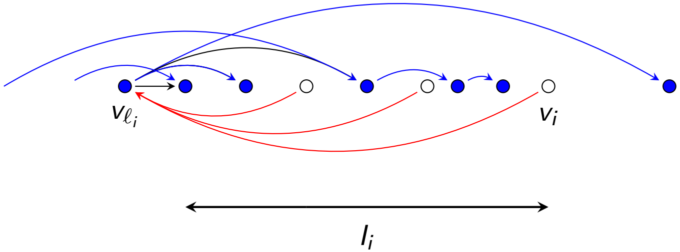
Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $L = \{\text{out-leaves}\}$.



$\text{unvd}(A) \le n + k - 1$: analysis

vertex v_ℓ is active for i if $v_\ell = \phi(a)$ for node a and a has a son b that is not embedded in $\{v_1, \ldots, v_i\}$.
For $v_i \in F$, let ℓ_i be the largest index such that v_{ℓ_i} is active for i.
Set $I_i = \{v_{\ell_{i+1}}, \ldots, v_i\}$.

Claim 1: If $v_i \in F$, then $|I_i \cap F| \le |I_i \cap \phi(L)|$. $L = \{\text{out-leaves}\}$.



$\text{unvd}(A) \leq n + k - 1$: analysis

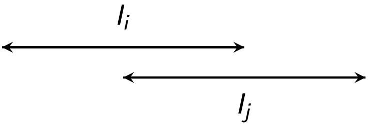
Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $L = \{\text{out-leaves}\}$.

Claim 2: If $v_i, v_j \in F$, then either $I_i \cap I_j = \emptyset$, or $I_i \subseteq I_j$, or $I_j \subseteq I_i$.

$\mathrm{unvd}(A) \leq n + k - 1$: analysis

Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $L = \{\text{out-leaves}\}$.

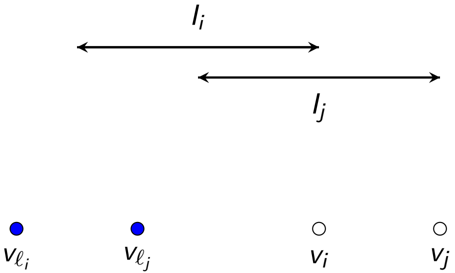
Claim 2: If $v_i, v_j \in F$, then either $I_i \cap I_j = \emptyset$, or $I_i \subseteq I_j$, or $I_j \subseteq I_i$.



$\mathrm{unvd}(A) \leq n + k - 1$: analysis

Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $L = \{\text{out-leaves}\}$.

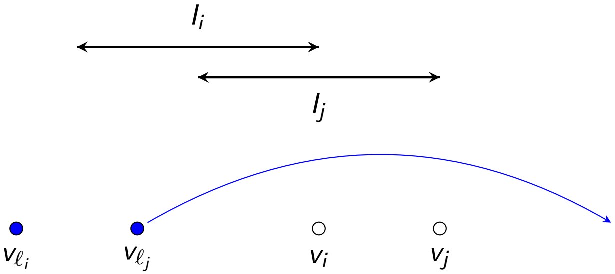
Claim 2: If $v_i, v_j \in F$, then either $I_i \cap I_j = \emptyset$, or $I_i \subseteq I_j$, or $I_j \subseteq I_i$.



$\text{unvd}(A) \leq n + k - 1$: analysis

Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $\quad L = \{\text{out-leaves}\}$.

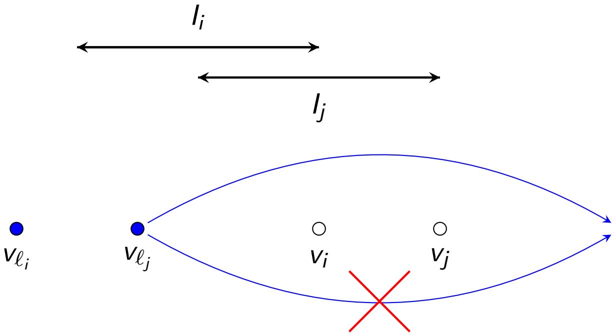
Claim 2: If $v_i, v_j \in F$, then either $I_i \cap I_j = \emptyset$, or $I_i \subseteq I_j$, or $I_j \subseteq I_i$.



unvd(A) $\leq n + k - 1$: analysis

Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $L = \{\text{out-leaves}\}$.

Claim 2: If $v_i, v_j \in F$, then either $I_i \cap I_j = \emptyset$, or $I_i \subseteq I_j$, or $I_j \subseteq I_i$.



unvd$(A) \leq n + k - 1$: analysis

Claim 1: If $v_i \in F$, then $|I_i \cap F| \leq |I_i \cap \phi(L)|$. $\quad L = \{\text{out-leaves}\}$.

Claim 2: If $v_i, v_j \in F$, then either $I_i \cap I_j = \emptyset$, or $I_i \subseteq I_j$, or $I_j \subseteq I_i$.

M: the set of indices i such that $v_i \in F$ and I_i is maximal for inclusion.
$|F| = \sum_{i \in M} |I_i \cap F| \leq \sum_{i \in M} |I_i \cap \phi(L)| \leq |\phi(L)| \leq k - 1$.

Thank you for your attention.

Questions ???