Forbidden trees and the beautiful trees of Grenoble

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Hungary

χ-bounded classes of graphs

Undirected version

Def

G is H-free if $\exists$ no H as induced sg in G

Forb H := { H-free graphs }

Forb *H* := { G | $\exists$ no H $\in$ *H* as induced sg in G }

B ---- class of graphs (mostly closed under taking induced sg's)

B χ-bounded:

$\exists$ f such that

for $\forall$G$\in$ *B* $\chi(G) \leq f[\omega(G)]$

($\exists$ f$\Rightarrow$ $\exists$ monotone f')

Oriented version

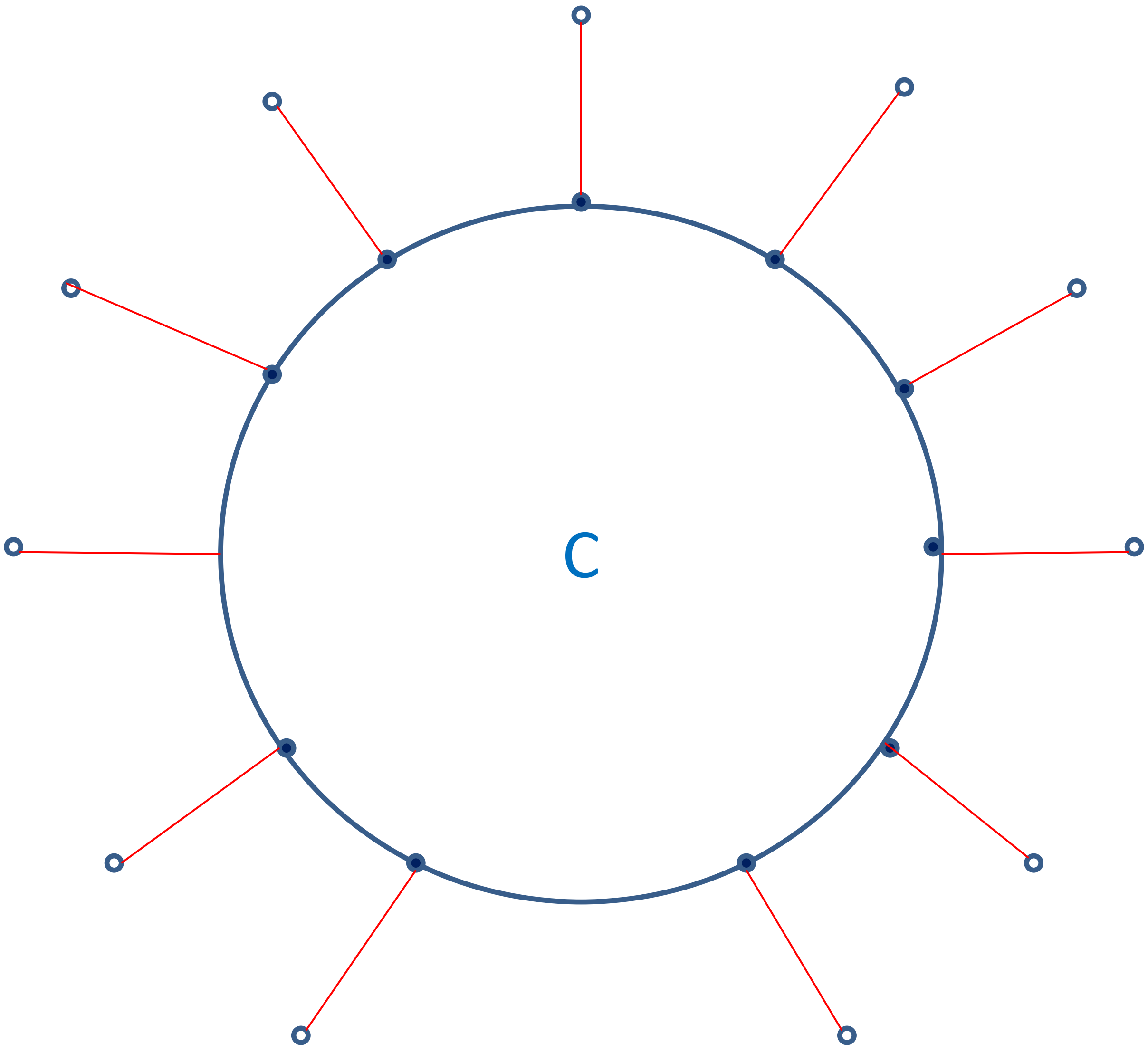
Induced subgraph → Induced subdigraph

[Or] χ-bounded families of oriented graphs,
Pierre Aboulker, Jørgen Bang-Jensen, Nicolas Bousquet, Pierre Charbit,
Frédéric Havet, **F** (Frédéric Maffray), Jose Zamora

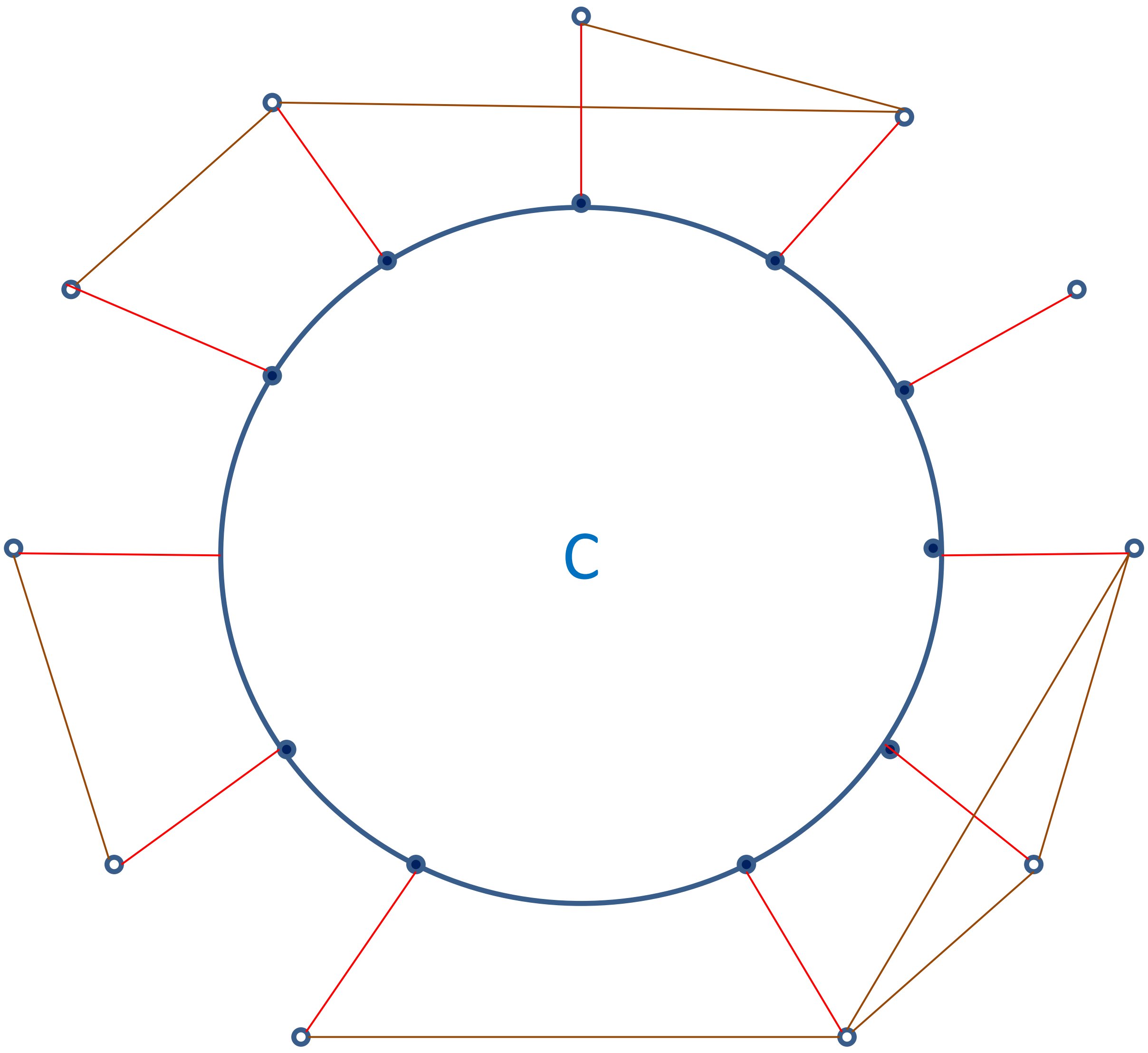
arXiv:1605.07411v1 [cs.DM] 24 May 2016

(On directed stars and orientations of P_4)

An open problem



Corona of (any) graph C



Generalized Corona of graph C

$GC := \{H \mid H$ is some generalized corona of $C_{2k+1}\}$

Open problem C

Is Forb GC χ-bounded?

Conjecture C : Yes.

Remark Conjecture C would imply [odd].

[BM] G. Bacsó, E. Boros, V. Gurvich, **F**, M. Preissmann:
On Minimal Imperfect Graphs with Circular Symmetry,
J.of Graph Theory, 29(1998) 210-225

A brief summary, using general concepts

Def

Γ--- (arbitrary) finite group

A,B$\subseteq\Gamma$ („complexes”)

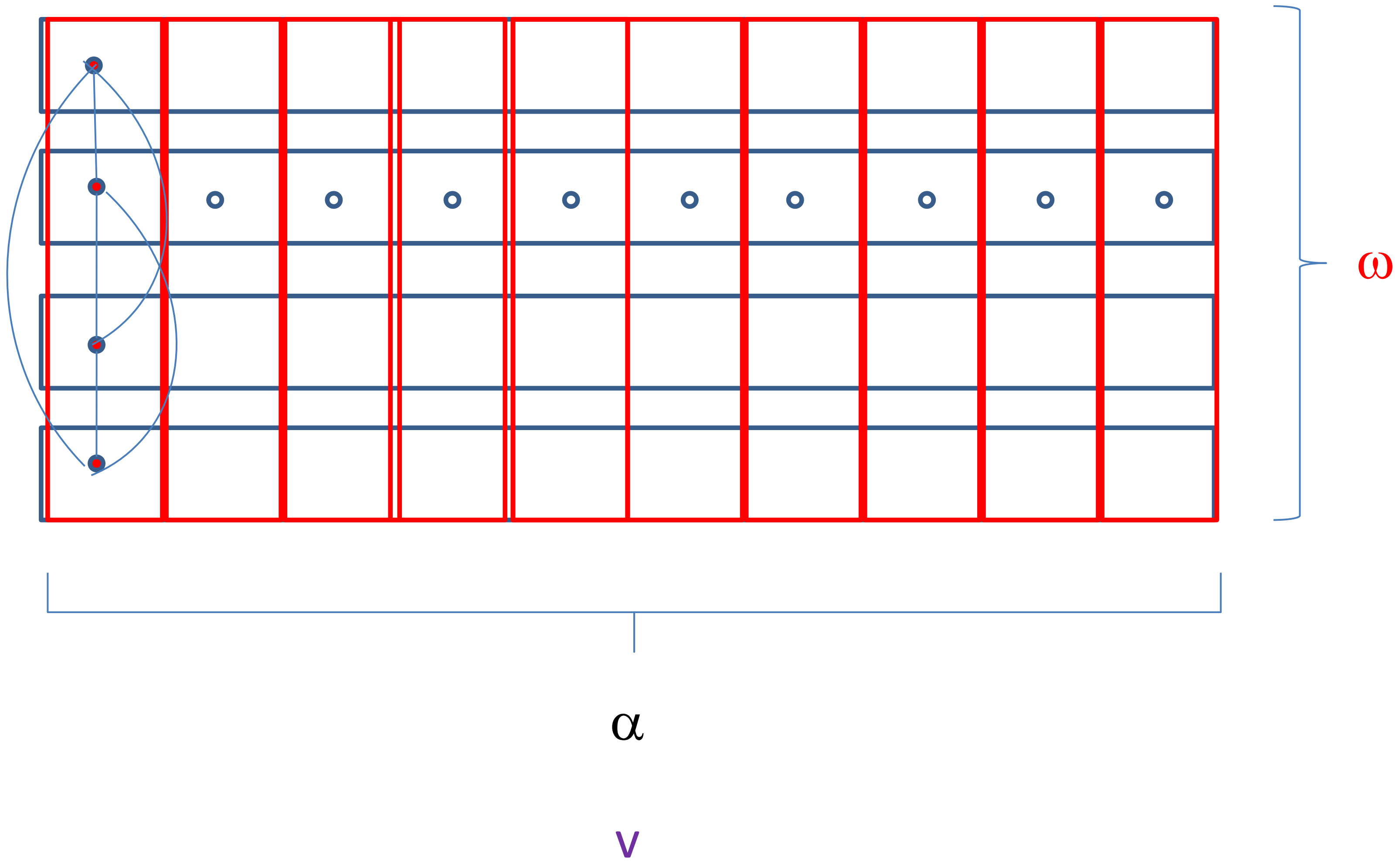
(A,B) near-factorization of Γ:

$\exists\nu\in\Gamma$ such that $\forall x\neq\nu$ $\exists!$ a$\in$A, b$\in$B with x=ab
and ν cannot be expressed so.

Short form: Γ-ν=AB

Def

Graph P partitionable: ∀v ∃ rectangle



Γ, near-factorization (A,B) of Γ $\rightarrow$P=P(A,B)

P:=(Γ,E),

$\mathcal{Q}$:= { xA | x$\in\Gamma$}

E:={uv| $\exists$Q$\in\mathcal{Q}$ such that u,v$\in$Q}

Proposition

P(A,B) is always partitionable.

Marriage of algebra and graph theory!

Lovász's theorem

$\forall$ Minimal imperfect graph is partitionable

But: Many non-min.-imp. partitionable graphs exist!

Examples

Web (except odd holes and anti-holes)

British Numeral System

In our paper the group Γ is generally Z_n

Open questions on near-factorizations

I. **Conjecture** [ae] Commutative but acyclic groups have no near-factorizations.

II. Some open questions in connection with graph theory, on α-critical edges, e.g.

On the same topics, but independently (~)

Maximal cliques of a graph

↓

Hypergraph

↓

Small chromatic number?

Main conjecture – refuted recently [IP]

P_5-free graphs, coloring problems

(1) Given a P_5-free graph G, $\chi(G)$=?
----NP-hard.[KrKr]

Given a fixed k, (1) can be solved in polynomial time. [Fix]

Theorem (**F**, Morel, G) [MM]
Linear time can also be realized.

Forb P_5 is an „easy class" in some sense.

But ----„Contradicting example": ω for $2K_2$-free graphs ($2K_2 \subseteq P_5$)

P_5-free graphs, [maximum] stable set problem

(independent set)

For this problem the class is „easier”

History

Def

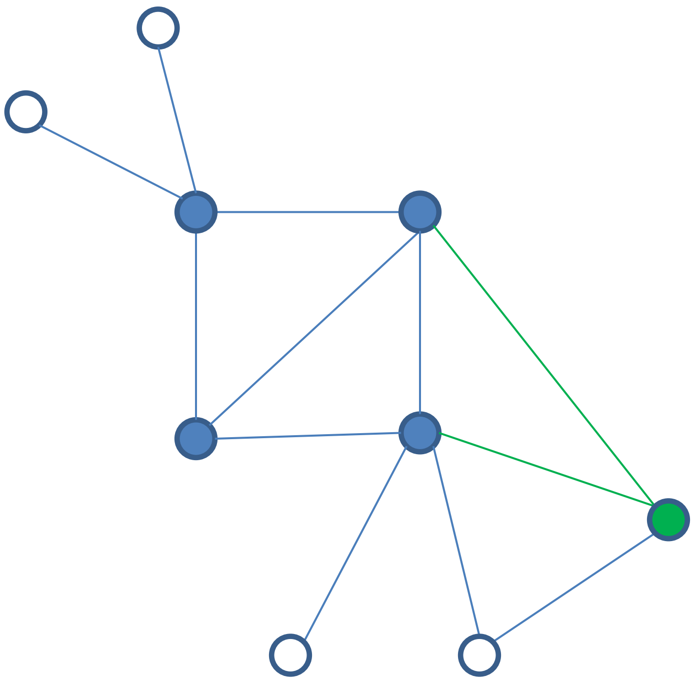
Subexponential algorithm:

f:=Runnning function = $O(\exp(n^{\varepsilon}))$, $\varepsilon<1$

Or $f=O[e^{o(n)}]$

Or…

Theorem (B. Randerath, I. Schiermeyer, 2010)
∃Subexponential algorithm (in the stronger sense) for the stable set problem in P5-free graphs .



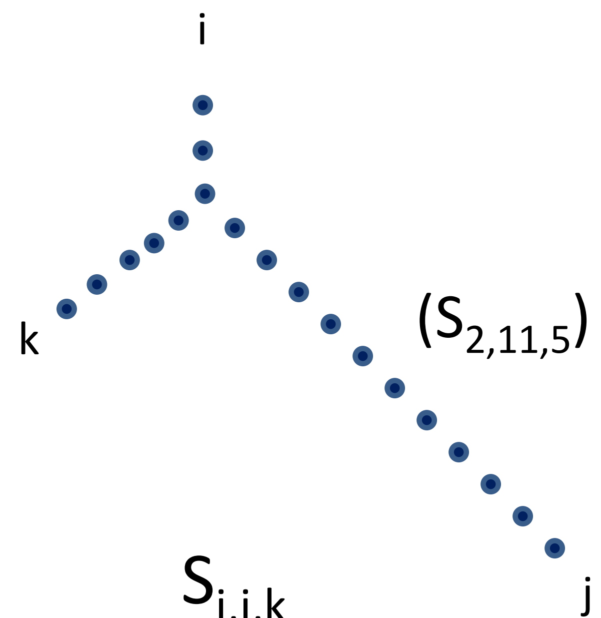
Def
Dominating set: Everything outside has at last one neighbour inside.

Remark In the result above,, a „harmless" lemma [BZS] has been applied:

Lemma
In a connected P_5-free graph, ∃ a dominating clique or a dominating P_3

$\alpha(G):=\max\{|T|: T$ stable set in $G\}$

H-problem: Compute α for H-free graphs (Briefly: $\alpha|H$)



$S_{i,j,k}$

Notation

$\mathcal{S}:=\{S_{i,j,k} | i,j,k=\ldots\}$

Theorem

H not in $\boldsymbol{S}$ $\Rightarrow$ $\alpha|H \in$ NPH

And the graphs in **S**?

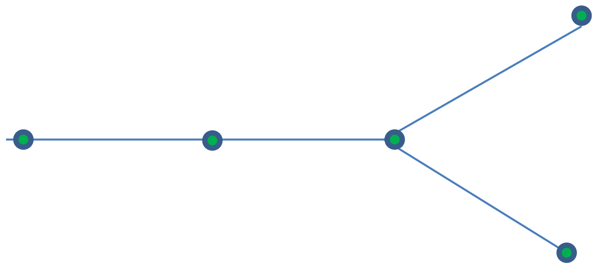
$H=P_t$

T=4 ---- obviously in P

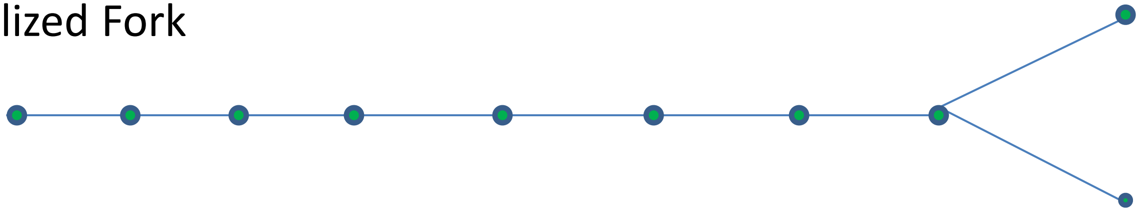
t=5,6 ---- in P [Lsh]

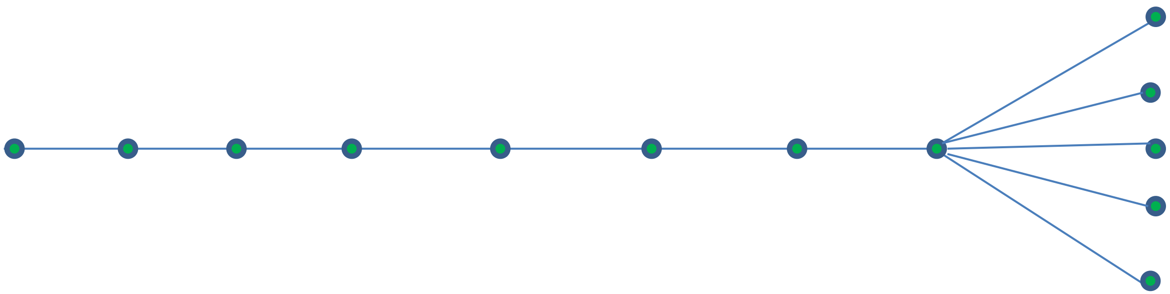
For any fixed t, $\exists$ subexponential algorithm for $\alpha|H$ [BDMZs]

H=Fork $(S_{1,1,2}) \Rightarrow \alpha|H \in P$ (Alekseev)



H --- Generalized Fork





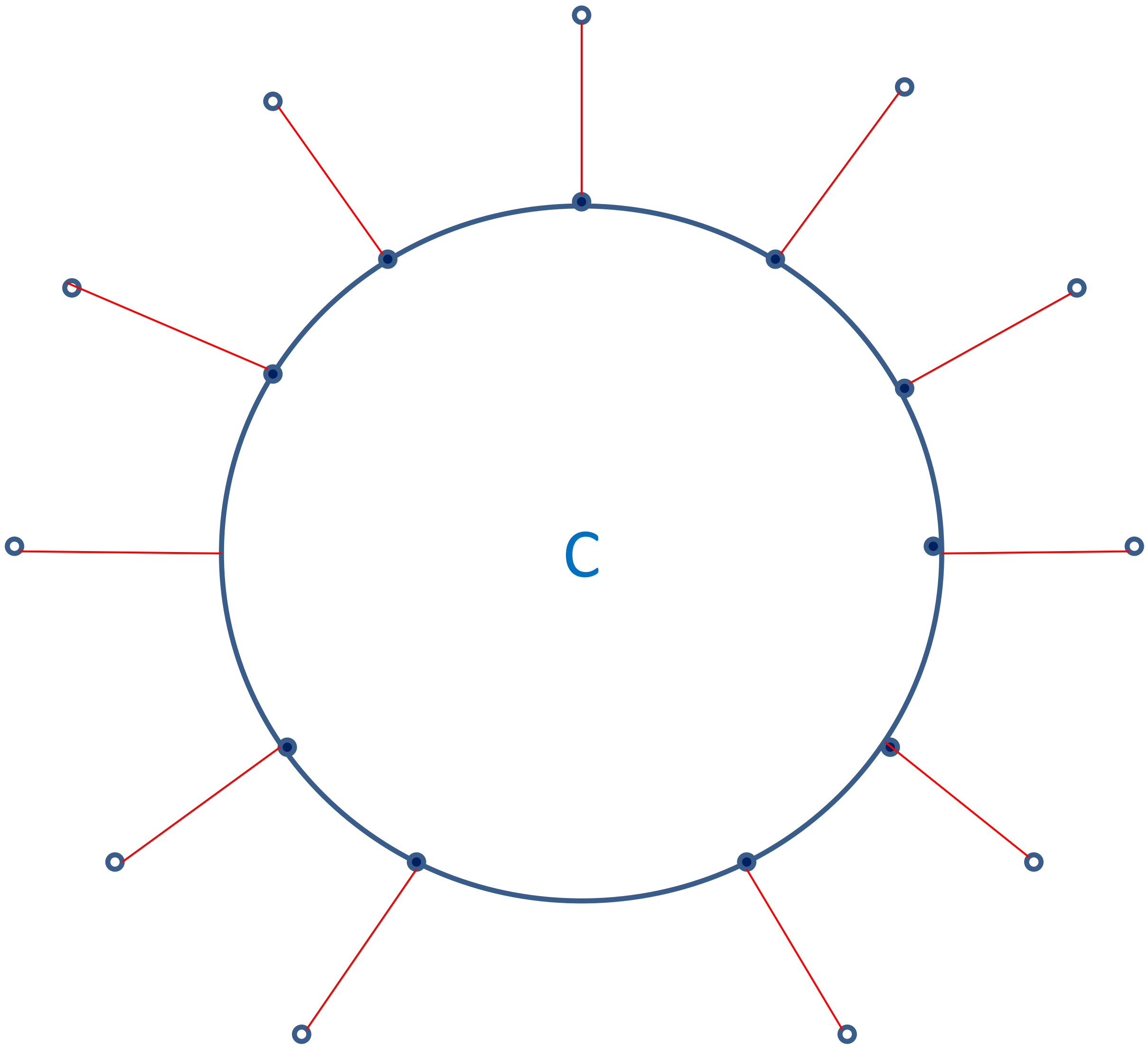
H ---Broom $\Rightarrow$ $\exists$ subexponential approximation algorithm [Six]

! Generally the Broom is not in ***S***

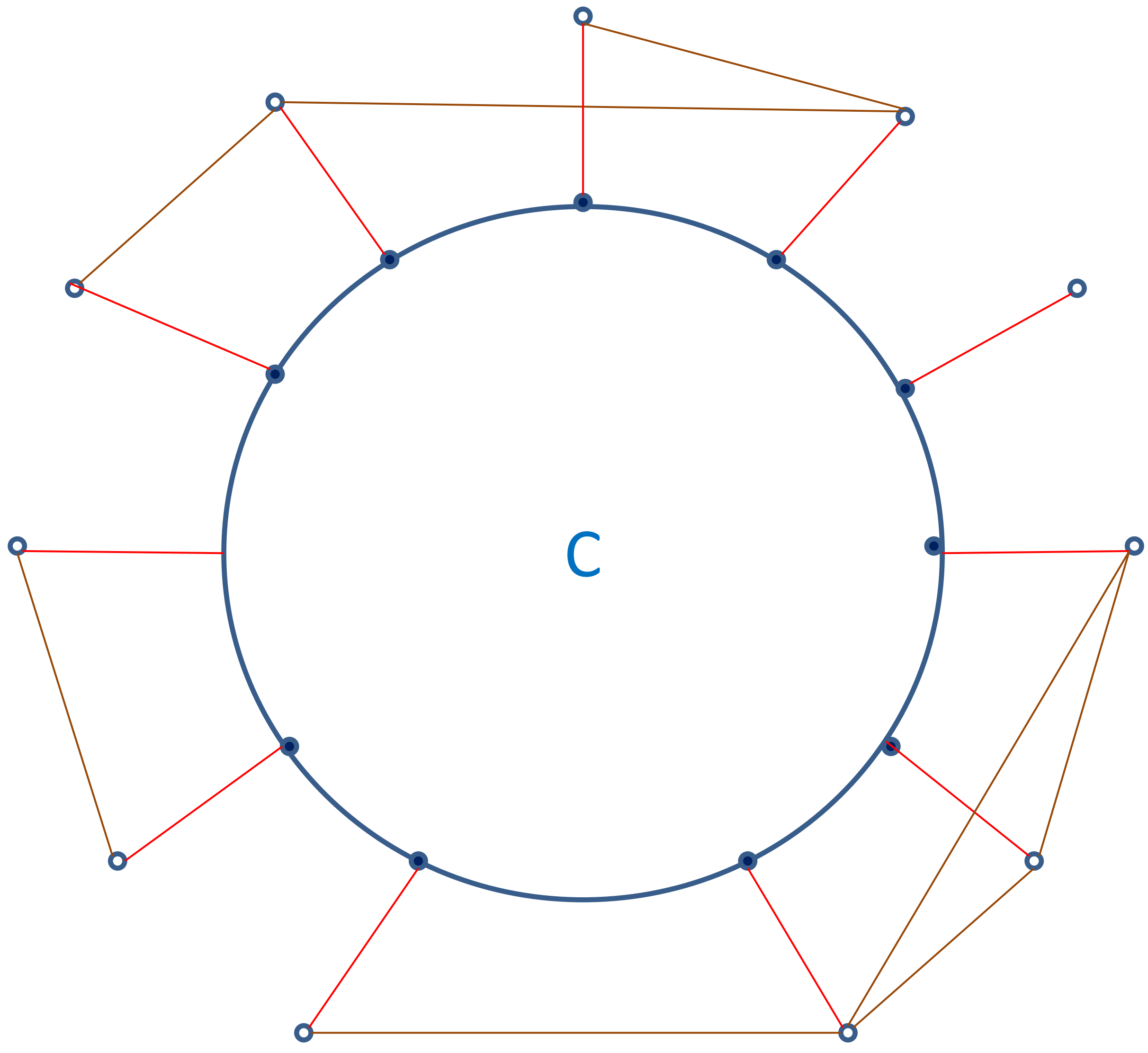
Even it is possible:

„$\forall H \in \boldsymbol{S}, \quad \alpha | H \in P$”
(But it will be very difficult to priove it.)

An open problem



Corona of (any) graph C



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The Gyárfás-Sumner Tree Conjecture (GSTC)

GSTC Given a fixed tree T, Forb T is χ-bounded

(For forests, reducible to trees)

Generalizing Conjecture C, we obtain Conjecture D…

Theorem (It has a proof!) Conjecture D $\Rightarrow$ GSTC

Conversely? --- ???

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[Or]
Pierre Aboulker, Jørgen Bang-Jensen, Nicolas Bousquet, Pierre Charbit, Frédéric Havet, **F** , Jose Zamora: χ-bounded families of oriented graphs, arXiv:1605.07411v1 [cs.DM] 24 May 2016

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I. Odd holes
2015

Thank you for your attention!