Odd pairs of cliques

Nicolas Trotignon
Univ Lyon, CNRS, ENS de Lyon
A tribute to Frédéric Maffray
Grenoble, Sep. 2019

Karthick Thiyagarajan

Berge graphs and perfect graphs

- A graph G is *perfect* if all its induced subgraphs G' satisfy

$$\chi(G') = \omega(G')$$

- A graph is *minimally imperfect* if it is not perfect but all its induced subgraphs are perfect
- Examples of minimally imperfect graphs:
chordless odd cycles of length at least 5, their complements (*odd holes* and *odd antiholes*)
- A graph is *Berge* if it contains none of these (as induced subgraph)

The strong perfect graph theorem

- The *SPGT*, proved by Chudnovsky, Robertson, Seymour and Thomas in 2002: every Berge graph is perfect
- *Rephrasing*: odd holes and odd antiholes are the only minimally imperfect graphs

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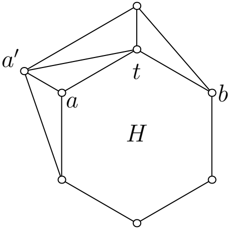

The Fonlupt and Uhry theorem

Fonlupt and Uhry, 1982:

- Let G be a graph, and a, b be an even pair in G
- Let G/ab be the graph obtained by contracting a and b to a single vertex
- Then:

$$\chi(G/ab) = \chi(G)$$

$$\omega(G/ab) = \omega(G)$$

So: even pairs contractions preserve being perfect.

Meyniel's theorem

Meyniel, 1987:

- A minimally imperfect graph has no even pair

Even pairs and the SPGC (1)

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- But it is false ...

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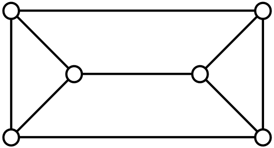

Even pairs and the SPGC (2)

- “For every Berge graph G on at least two vertices, G or $\overline{G}$ has an even pair” would imply the SPGT
  (because Lovász proved that G is perfect if and only if $\overline{G}$ is perfect).
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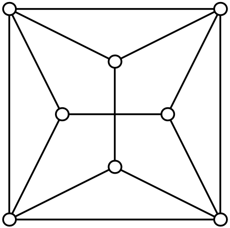

Even pairs in bull-free graphs

A theorem, the unique one of this kind, by de Figueiredo, Maffray and Porto 1997:
if G is Berge and bull-free, then either $|V(G)| = 1$, or G has an even pair, or $\overline{G}$ has an even pair.

Hourgardy's counter examples

Hougardy proved that infinitely many line graphs of bipartite graphs G, none of G and $\overline{G}$ contain an even pair

Hence: proving the SPGT purely with even pairs is seemingly hopeless...

Odd pairs of cliques

An *odd pair of cliques* in a graphs G is a pair K_1, K_2 such that:

- K_1 and K_2 are cliques in G
- Every chordless path P with one end in K_1, one end in K_2 and no internal vertex in $K_1 \cup K_2$ has odd length

Note: the cliques are disjoint.

Burlet's conjectures

Two conjectures:

- If G is Berge, then G or $\overline{G}$ contains an even pair or an odd pair of maximal cliques
- If G is minimally imperfect, then G contains no odd pair of maximal cliques

This would imply the SPGT

Results

Results proved with Frédéric Maffray in 2001
(independently by András Sebő):

- If G is perfect and contains an odd pair of cliques K_1, K_2 then $G_{K_1 \equiv K_2}$ is perfect
- If G is a basic perfect graph (bipartite, line graph of bipartite, complement of these, or double split graph), then G or $\overline{G}$ contains an even pair or an odd pair of maximal cliques
- A minimally imperfect does not contain a odd pair of maximal cliques K_1, K_2 such that $|K_1| + |K_2| = \omega(G)$

My last joint work with Frédéric

With Maria Chudnovsky and Kristina Vušković: coloring square-free perfect graphs in polytime.