The Chain Graph Sandwich Problem

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Joint work with :

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Frederic Maffray (L. G-Scop, France).

CAPES / COFECUB

Academic cooperation agreement between Brazil and France

1997-2000   Luiz Satoru / Claudia L. – Frederic Maffrey

2001-2005   Celina de Figueiredo – Frederic Maffrey

2008-2011   Sulamita Klein – Zoltan Szigeti
2014

Graph Sandwich Problem

GRAPH SANDWICH PROBLEM FOR PROPERTY Π

Instance: $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$, $E^1 \subseteq E^2$

Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$

that satisfies property Π ?

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In a **sandwich problem**, we look for a third graph, whose edge set lies between the edge sets of two given graphs. This third graph is required to satisfy a property Π.

Observations:

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If the property Π is **ancestral**, then there exists a **sandwich graph** for (**V**, E^1, E^2) with the property Π, if and only if G^2 = (**V**, E^2) has the property Π.

Solved classes of graphs

Graph Sandwich Problems in **P**:

Tree
Bipartite
Split graphs (M. C. Golumbic, H. Kaplan, R. Shamir , 1995)
Cographs
Threshold
(k,l)-graphs, $k+l<3$ (Dantas, de Figueiredo, Faria, 2004)
P4-sparse graphs (Dantas, K., Mello, Morgana 2008)

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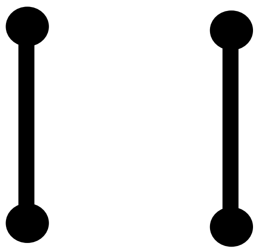
Graph Sandwich Problems that are **NP-complete**:

Comparability graphs
Permutation graphs
Chordal graphs
Interval graphs (Golumbic, Kaplan, Shamir, 1995)
Proper interval graphs
Undirected Path
Directed Path
(k,l)-graphs, k+l >=3 (Dantas, de Figueiredo, Faria, 2004)
Strongly chordal (de Figueiredo, Faria, K., Sritharan, 2007)
Chordal bipartite (Sritharan, 2008)

Chain Graphs

A **chain graph** is a **$2K_2$-free** bipartite graph. (Yannakaks 1981)



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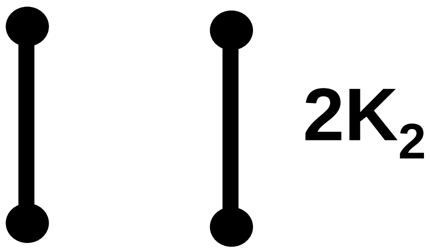
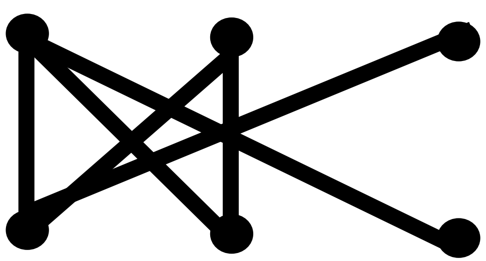


Chain graphs are also known as **difference graphs**.

(Hammer, Peled and Sun 1990)



Chain Graphs x Threshold Graphs

A graph is a **threshold graph** if its vertex set can be partitioned into a clique **K** and an independent set **S** such that any two vertices in **S** have inclusionwise comparable neighbourhoods.

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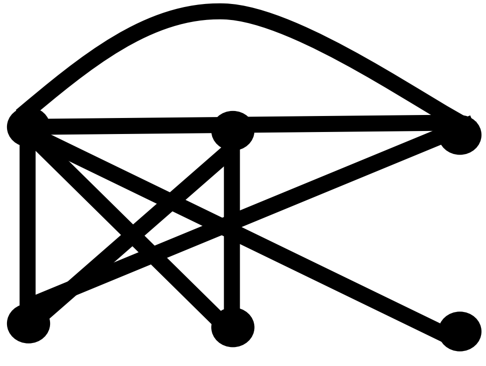

Threshold Graph

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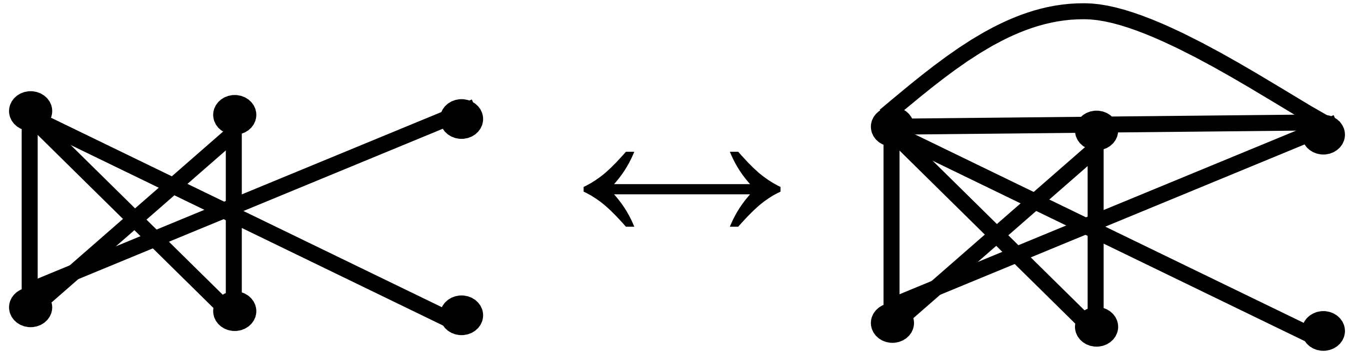

Chain Graph

Threshold Graph

Theorem: (Golumbic, Kaplan, Shamir, 1995)
The threshold sandwich graph problem is polynomial.

Partitioned Chain Probe Graph Problem

PARTITIONED CHAIN PROBE GRAPH PROBLEM

Instance: **G** = (**V**, **E**) and **a probe partition V=(N,P),** where **N** is an independet set.

Question: Is there a set **E'** $\subseteq$ **NxN** such that **H=(V, EUE')** is a **chain graph**?

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Theorem (Golumbic, Maffray, Morel 2009) (Van Bang 2010):

The **partitioned chain probe graph problem** is polynomial.

Chain Graph Characterization

Theorem (Mahadev and Peled 1995)

Let $\mathbf{G} = ((\mathbf{L},\mathbf{R}),\mathbf{E})$ be a bipartite graph, and let Z be the set of isolated vertices in $\mathbf{G}$. Then $\mathbf{G}$ is a **chain graph** if and only if either $Z = \mathbf{L} \cup \mathbf{R}$ or there exist an integer $h \geq 1$ such that $\mathbf{L} \setminus Z$ and $\mathbf{R} \setminus Z$ can be partitioned into non empty-sets $\mathbf{L_1},\ldots, \mathbf{L_h}$ and $\mathbf{R_1},\ldots, \mathbf{R_h}$ respectively in such way that two vertices $x \in L_i$ and $y \in L_j$ are adjacent if and only if $i+j \leq h+1$.

Chain Graph Characterization

j \ i	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

$h=4$

$h+1=5$

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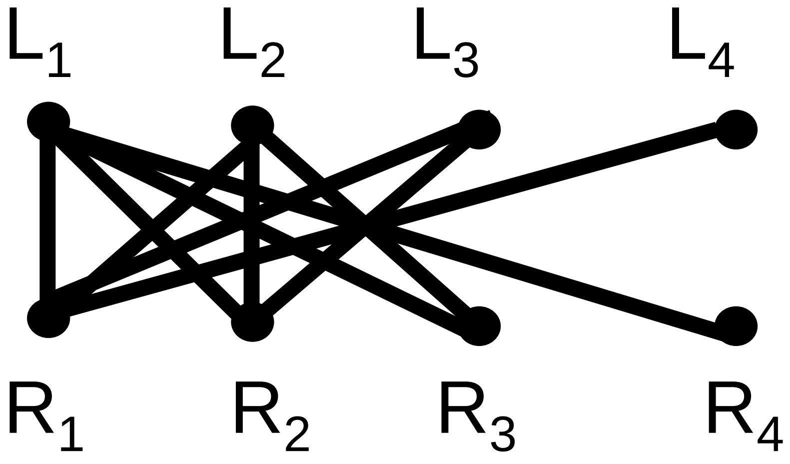

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Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G is **chain graph**?

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Question: Is there a graph $G = (\mathbf{V}, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G is **chain graph**?

Theorem: The **chain graph sandwich problem** is NP-Complete, even when restricted to the class of instances $(\mathbf{V}, E^1, E^2)$ where E^1 is a matching.

Proof (sketch):

The problem is in NP, as we can exhibit a graph which is a solution of the problem and check in polynomial time that it is a **chain graph.**

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NOT-ALL-EQUAL MONOTONE 3-SATISFIABILITY (NAE MONO 3-SAT)

Instance: A Boolean function *f* in conjunctive normal form, given as the conjunction of clauses $C_1, \ldots, C_m$ over a set $\{x_1,\ldots,x_n\}$ of variables, where each clause has three literals and no negative literals.

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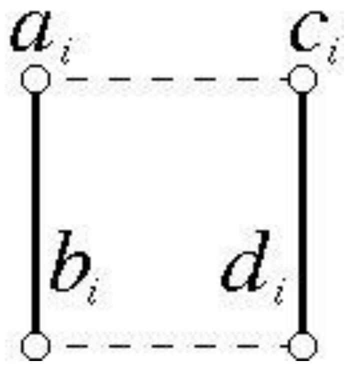
Question: Is there a truth assignment of the variables $x_1,\ldots,x_n$ such that each clause of *f* has at least one true literal and at least one false literal?

Construction of an instance of Chain Graph Sandwich



Truth Setting
Component
(Variable Gadget)

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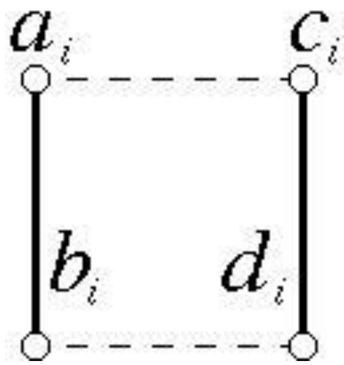

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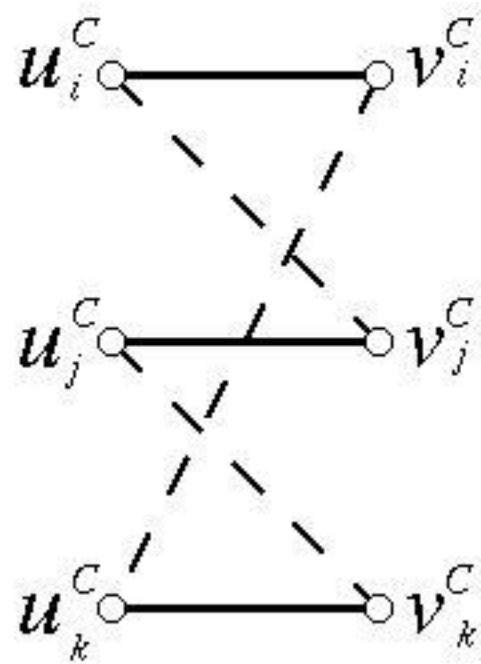
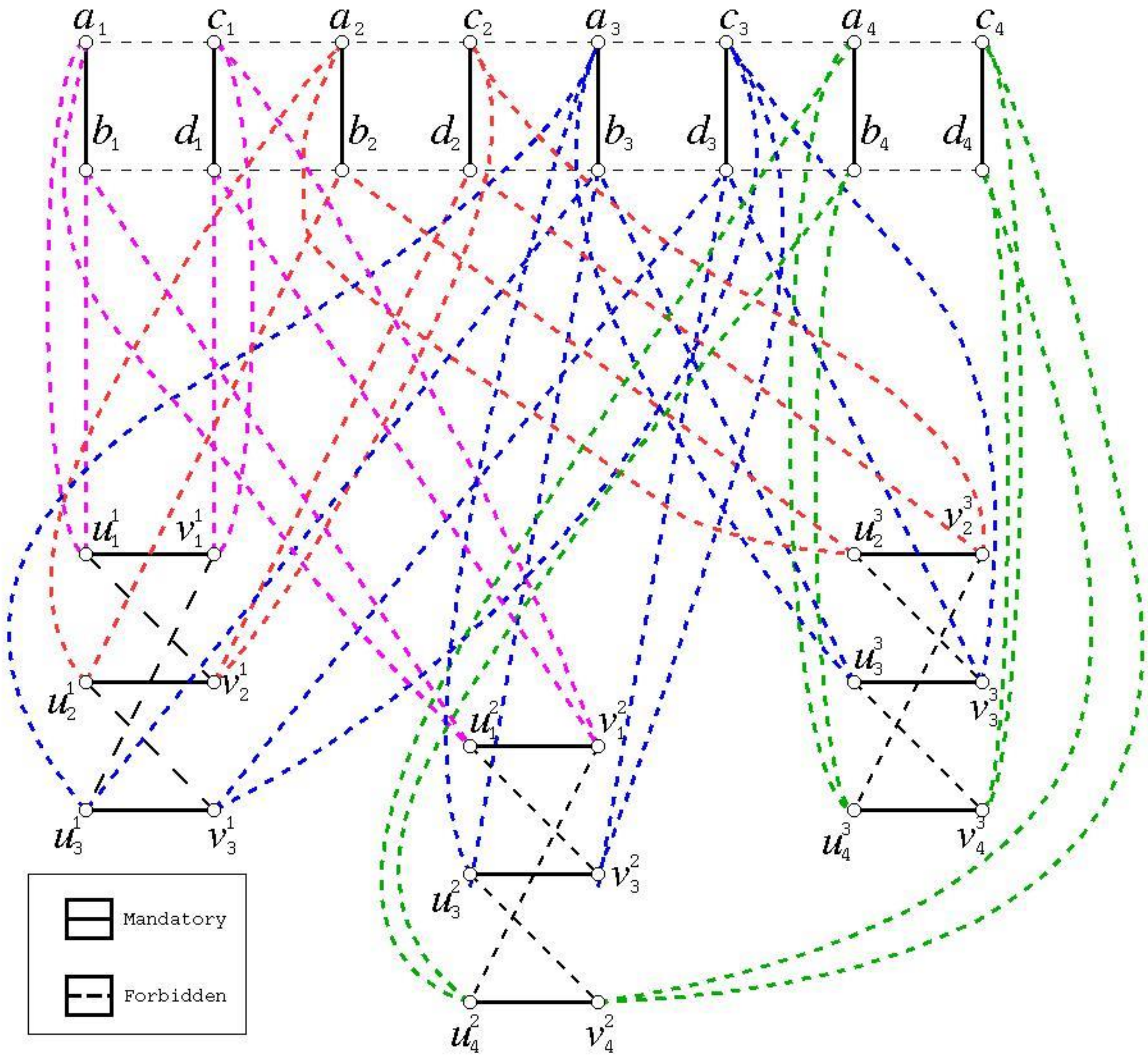
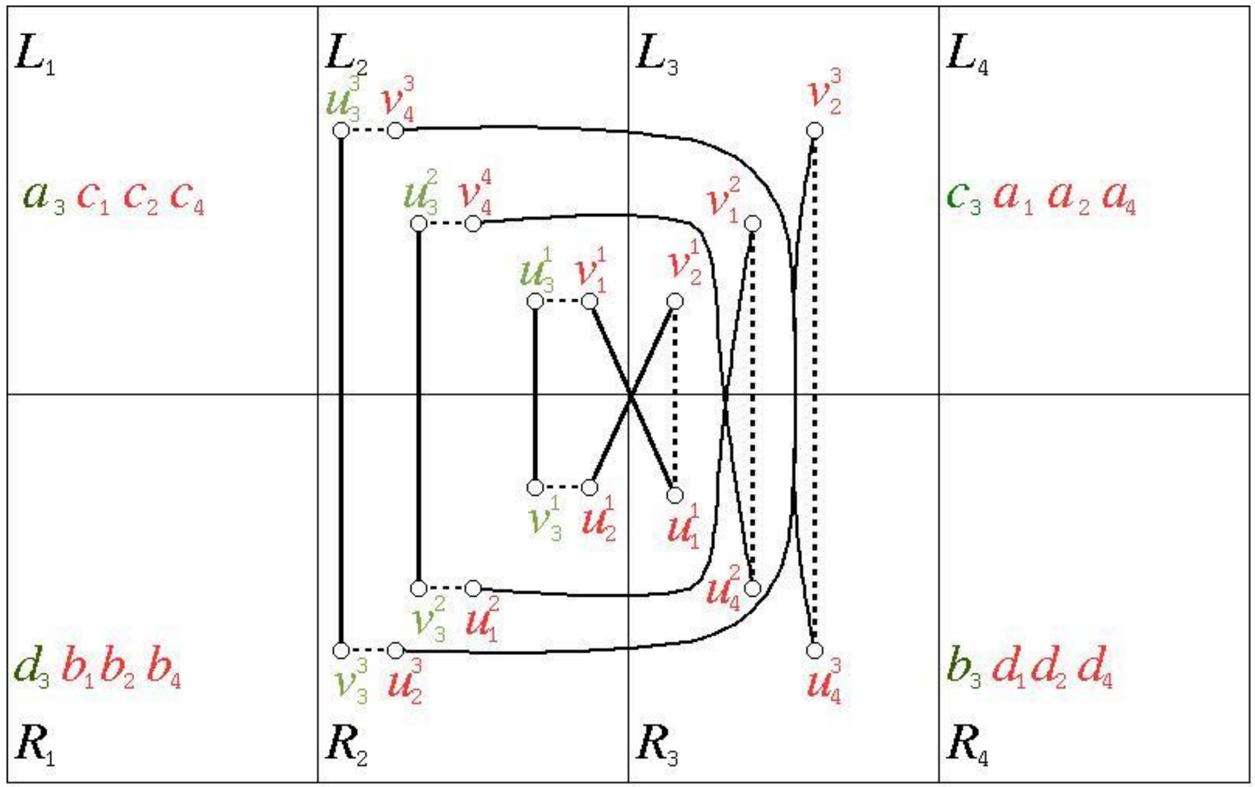
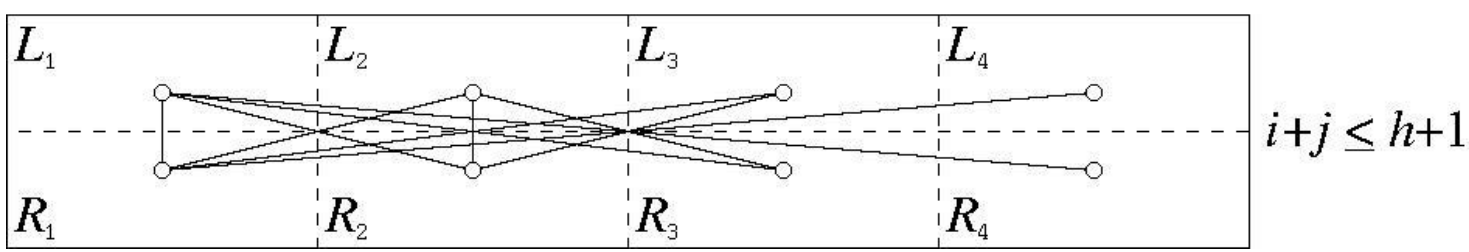


Satisfaction Testing (Clause Gadget)



(G^1,G^2) chain sandwich instance obtained from the 3SAT NAE-mono instance:
$I=(U,C)=(\{x_1, x_2, x_3, x_4\},\{(x_1, x_2, x_3), (x_1, x_3, x_4), (x_2, x_3, x_4)\})$.





Chain sandwich graph G for instance (G^1,G^2), obtained from the NAE-mono satisfiable truth assignment $x_1 = x_2 = x_4 = \mathrm{F}$, and $x_3 = \mathrm{T}$.

Final Observations

We have proven that the **CHAIN GRAPH SANDWICH PROBLEM** is NP-complete.

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The reason behind this high computational complexity lies in the observation that the bipartition of the vertices in any potencial **chain sandwich graph** solution is not fixed in advance.

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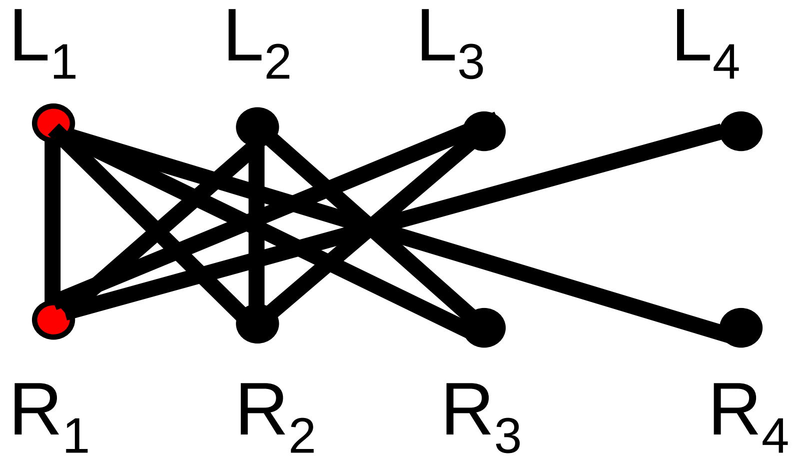
When $\mathbf{G^1}$ has only one non-trivial connected component, or, more generally, if the bipartition is fixed the **chain graph sandwich problem** can be solved in linear time, as implied by (Golumbic 2009).

Another characterization of chain graphs

Theorem: (Mahadev and Peled 1995)
A bipartite graph is a **chain graph** if and only if every induced subgraph has at most one non-trivial component **H=((L,R), F)**, and **H** has a vertex x in **L** universal to **R** and a vertex y in **R** universal to **L**.



Algorithm

1. Test the non-trivial component of G^1 to see that it is bipartite. If it is not bipartite there is no chain sandwich solution. If it is bipartite, then fix this bipartition-arbitrarily place the isolated vertices on any side.
2) Initialize $X \leftarrow V \quad E \leftarrow E^1$
3) Repeatedly, either remove an isolated vertex z from (X, E_X^1)

$$X \leftarrow X - \{z\}$$

or remove a vertex x universal to all vertices on the other side of the bipartition in (X, E_X^2) and add to E all of these edges incident on x:

$$X \leftarrow X - \{x\} \qquad E \leftarrow E \cup \{x,y \mid y \in N_X(x)\}$$

4) If the graph can be reduced to the empty graph, them there is a chain graph solution, namely, the final value of $G=(V,E)$; otherwise, there is not.

GT-2004 Conference in memory of Claude Berge

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ICGT-2005 International Colloquium in Graph Theory


EXIT

ICGT-2018 International Colloquium in Graph Theory

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