Easily Testable Graph Properties

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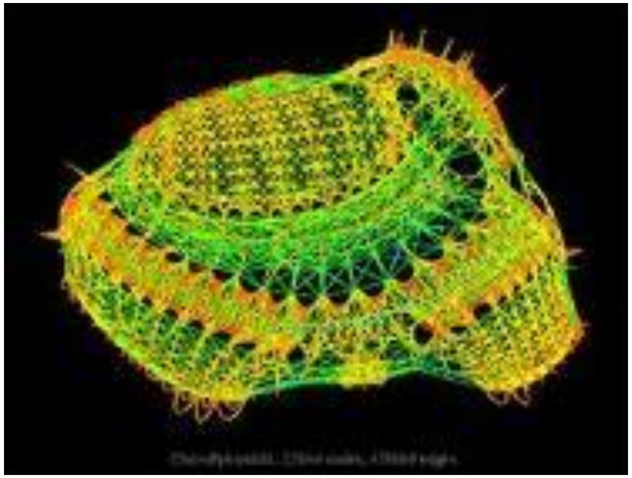
Joint work with Jacob Fox



Grenoble, 2014

Property Testing: a (very informal) motivation

Graph Property Testing

Objective [Goldreich, Goldwasser, Ron (96)]: distinguish between graphs on n vertices that satisfy a property P, and ones that are ε-far from satisfying it, by inspecting the induced subgraph on a random sample of only f(ε) vertices.

Here ε-far means that one has to delete or add to the graph at least εn^2 edges to get a graph satisfying the property.

A graph property is called testable if a random sample of f(ε) vertices suffices.

We consider 1-sided testers, that is, if the graph satisfies P, we will surely say so by inspecting its sample, if it is ε-far from satisfying P, the sample will reveal, with probability at least 0.9, that it does not satisfy P.

Intuitively, a (global) property is testable iff it can be inferred from local information.

Which graph properties are testable ?

A property is hereditary if it is closed under taking induced subgraphs.

Examples: 3-colorable, H-free, Perfect, Planar, Comparability, Chordal, Interval, Intersection graph of boxes in R^{17} , having a 2-edge coloring with no monochromatic triangle, …

Not hereditary: disjoint union of two isomorphic graphs, decomposable into edge-disjoint triangles, decomposable into Hamilton cycles,…

Thm (A-Shapira 05):

Any hereditary property is testable. The converse is also (essentially) true:

A graph property is testable (by an oblivious tester) iff it is (semi-) hereditary

Intuition: a sample leads to the conclusion that G does not satisfy P iff it is forbidden as an induced subgraph

The proof is based on a strong version of Szemerédi's Regularity Lemma [A, Fischer, Krivelevich, M. Szegedy (00)].

Lovász, B. Szegedy(06): a subsequent alternative proof based on convergent graph sequences

Remark 1: testability implies the local nature of hereditary graph properties: if G is ε-far from satisfying a hereditary property P, then G contains a small witness (for not satisfying P): an induced subgraph on at most f(ε) vertices that does not satisfy P.

(For P=k-colorability this was proved by Bollobás, Erdős, Simonovits and Szemerédi (78) for k=2 and by Duke and Rödl (85) for every k)

Remark 2: The upper bounds obtained for f(ε) are huge even for seemingly simple properties, like being perfect [Tower of height $1/\varepsilon^{O(1)}$ or worse]

Easily Testable Properties

Def: a graph property is called **easily testable** if it is testable with samples of size

$$f(\varepsilon) \leq (1/\varepsilon)^{O(1)} = Poly(1/\varepsilon)$$

Question: what are the easily testable graph properties ?

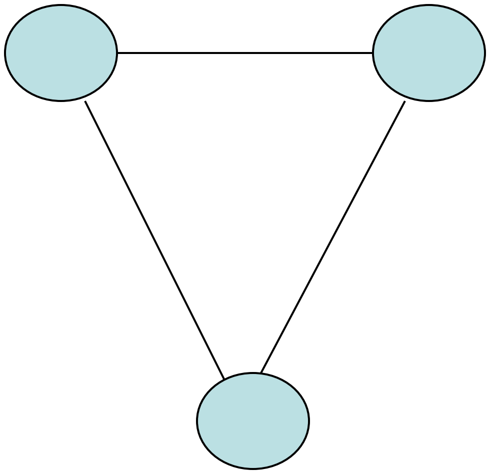
Previous results:

Being k-colorable is easily testable
[implicit in Duke-Rödl (85),
A-Duke-Lefmann-Rödl-Yuster (92),
explicit in Goldreich-Goldwasser-Ron(96),
improved bounds in A-Krivelevich (02),
Sohler (12)]

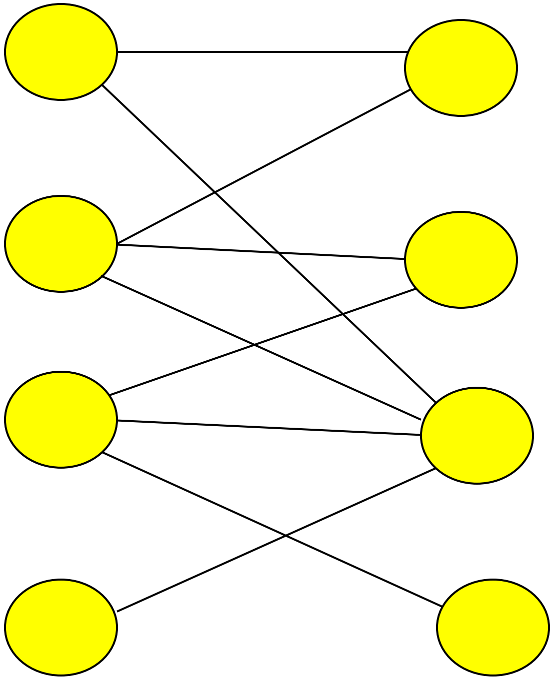
Goldreich-Trevisan(03): Similar more general "partition problems" are easily testable.

Thm (A-00): For a fixed graph H, the property P_H of being **H-free** is easily testable if and only if H is **bipartite**.

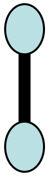
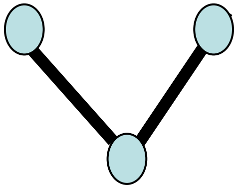
H_1 : not easy



H_2 : easy



Thm [A-Shapira (06)]: The property P^*_H of being induced H-free is easily testable for

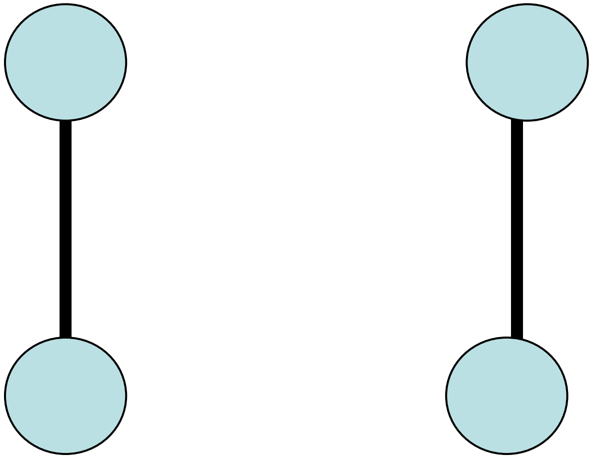
H=P_1 =  , for H=P_2= 

and for their complements, maybe it is easily testable for H=P_3 , H=C_4 and their complements, and it is not easily testable for any other H.

Thm 1 [A-Fox(14)]: **The property of being induced H-free for H=P_3 is easily testable**

Note: this is the property of being a cograph, a theorem of Seinsche(74) gives the structure of these graphs: these are all graphs generated from the single vertex by complementation and disjoint union.

Example:



What about perfectness ?

The Strong Perfect Graph Thm [Chudnovsky, Robertson, Seymour, Thomas (06)]: A graph is perfect iff it contains no induced odd cycle on at least 5 vertices (odd hole) or the complement of one (odd antihole).

This is proved by establishing a strong structural theorem for these graphs

Thm 2 [A-Fox(14)]: **The property of being perfect is not easily testable.**

Similarly: the property of being a comparability graph is not easily testable.

Something about the proofs

Prop 1: being induced P_2 –free is easily testable.

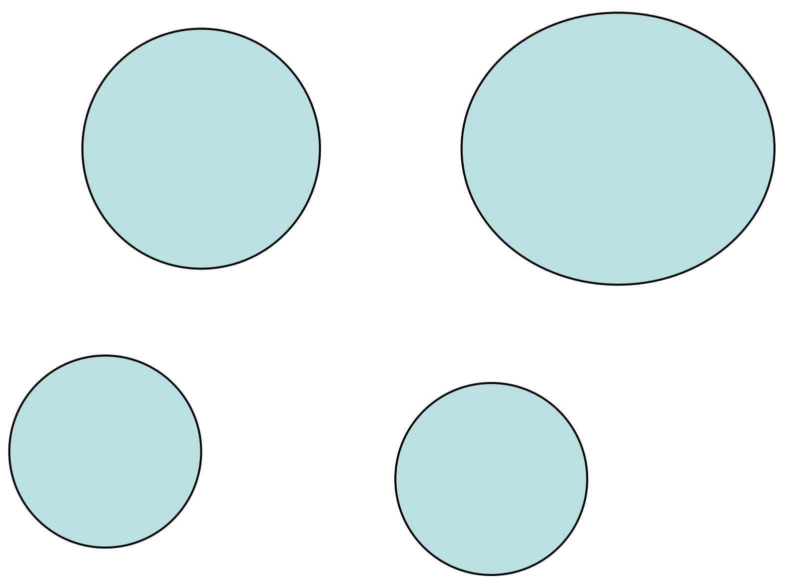
Proof (sketch): a graph is induced P_2 –free iff it is a vertex disjoint union of cliques.

Suppose G=(V,E) on n vertices is ε-far from being such a union. We show that an induced subgraph on a random set of O(1/ε log (1/ε)) vertices is not likely to be such a union.

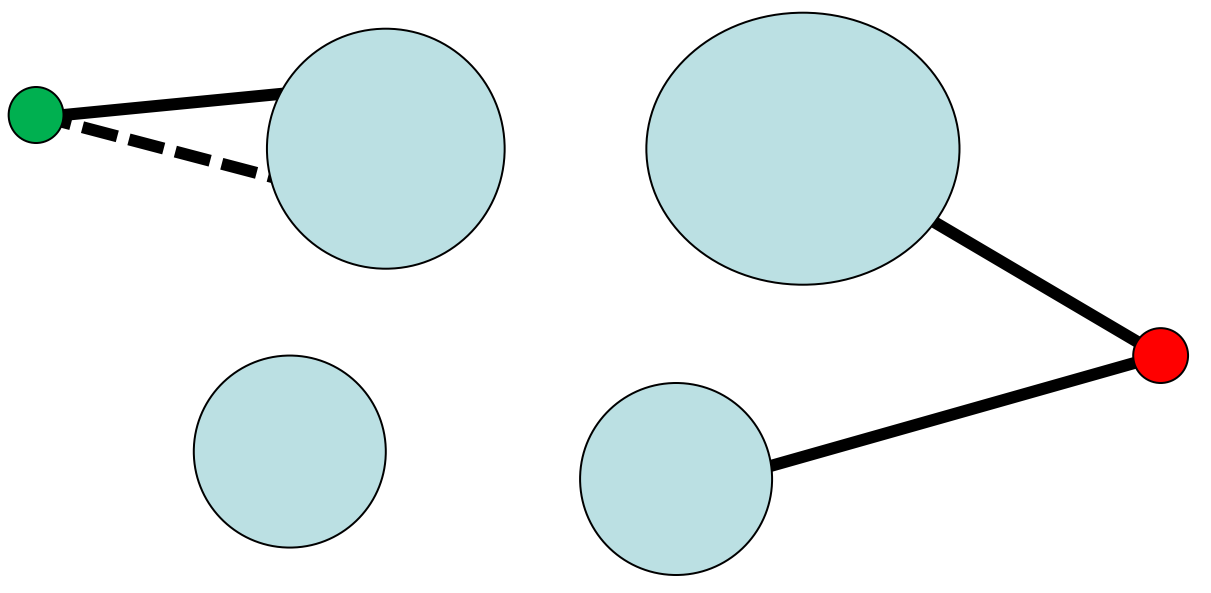
Choose this set in two steps, starting with a random set X and adding a random set Y.

May assume that each degree is at least εn/10 and that the induced subgraph on X is a disjoint union of cliques.

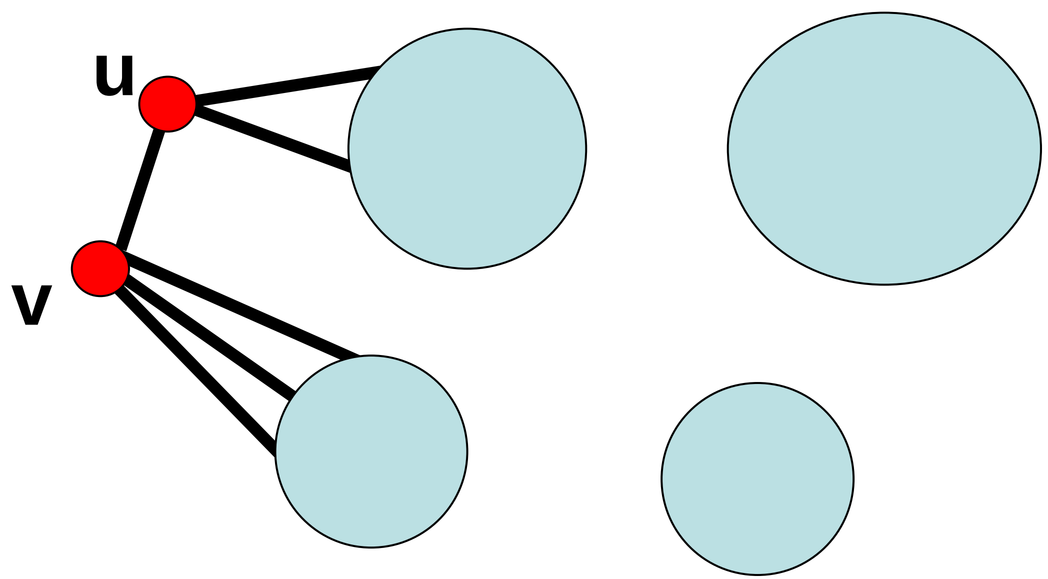
With high probability (whp) all vertices but εn/10 have neighbors in X



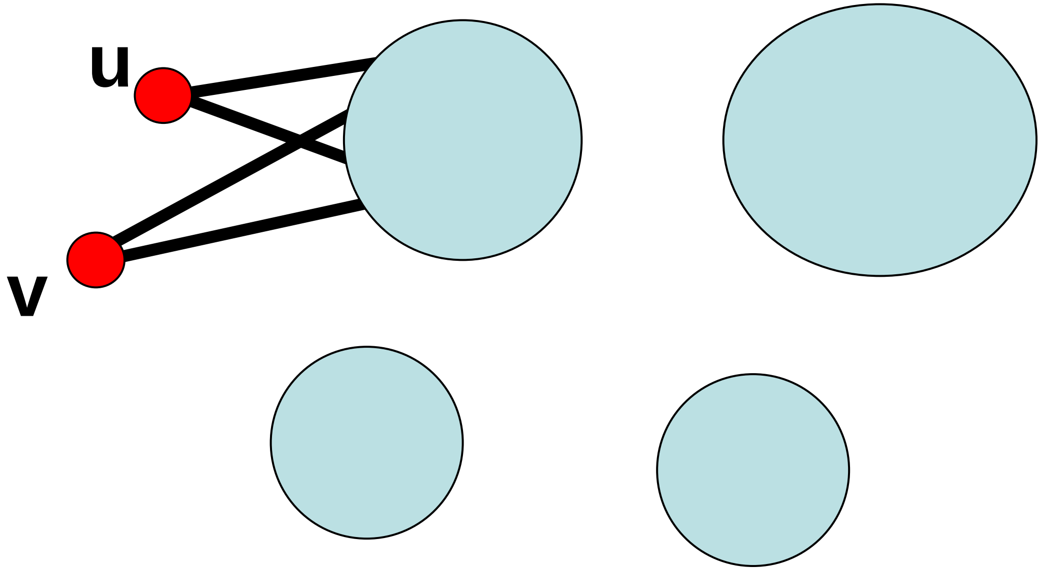
If there are at least εn/10 vertices having neighbors in two cliques of X, or having neighbors in some clique of X but without being connected to all of it, we'll find such vertex in Y, whp.



If there are at least $\varepsilon n^2/10$ pairs of adjacent vertices u,v, with u having neighbors in one clique and v in the other, we'll find such a pair in Y, whp.



Similarly, if there are at least $\varepsilon n^2/10$ pairs of non-adjacent vertices u,v, both having neighbors in the same clique, we'll find such a pair in Y, whp.



If none of these happens, then G is not ε-far from being a union of disjoint cliques, as we can classify most of the vertices according to the unique clique in X to which they are connected and turn G into a disjoint union of cliques on the resulting classes by adding and deleting less than εn^2 edges.

This contradicts the assumption that G is ε-far from being a union of disjoint cliques.

Therefore, being induced P_2–free is
easily testable.

Note that the existence of a simple structural description of induced P_2-free graphs (as a disjoint union of cliques) is helpful in the proof.

A similar (though more complicated) proof works for showing that being induced P_3 –free is easily testable. Here, too, the structural description of such graphs as cographs is crucial.

The proof here gives that a sample of size

$$2\,(100/\,\varepsilon)^{16}$$

suffices.

Thm: being perfect is not easily testable

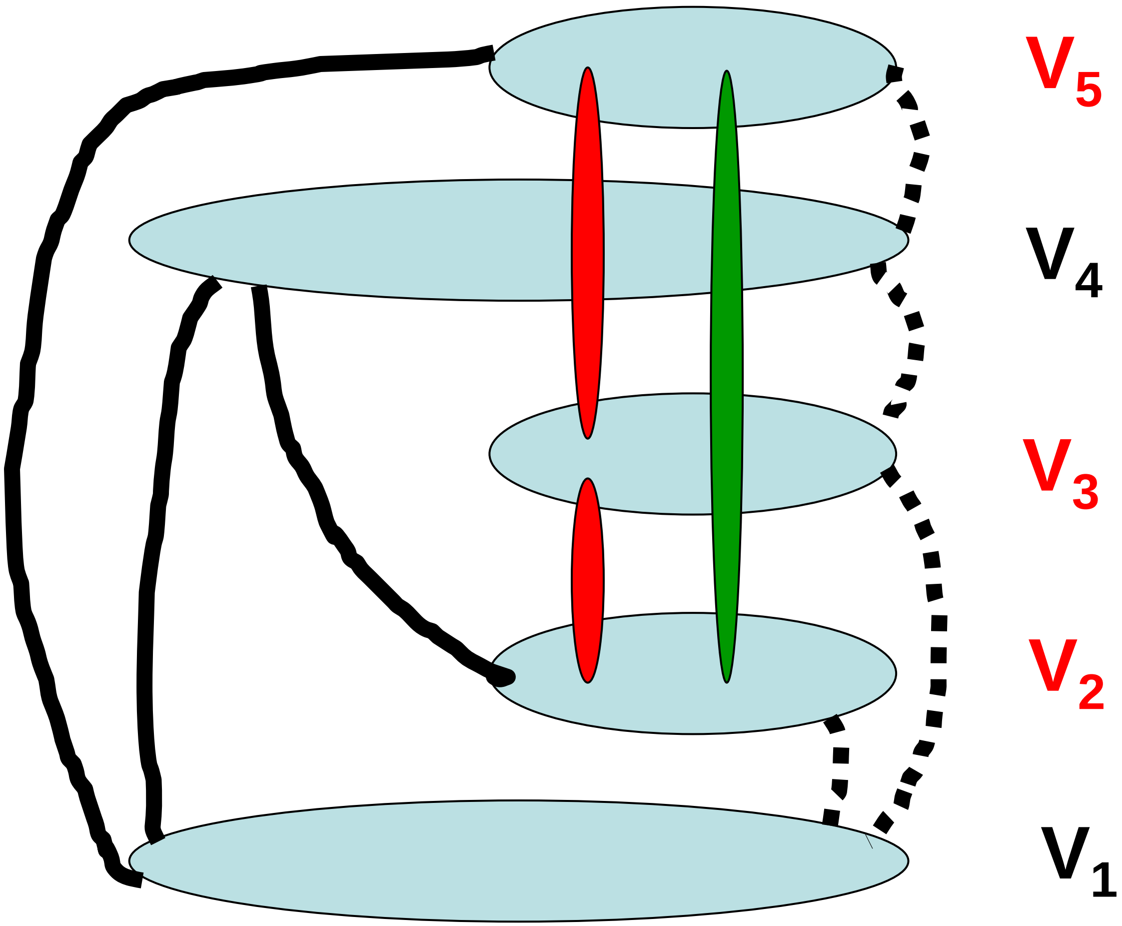
Proof (sketch):

Taking a blow-up of the graph of Ruzsa and Szemerédi (76), based on the construction of Behrend (46) of dense subsets of Z_n with no 3-term arithmetic progressions we get the existence of a 3-partite graph F with n/3 vertices in each vertex class, which is ε-far from being triangle-free, but contains only

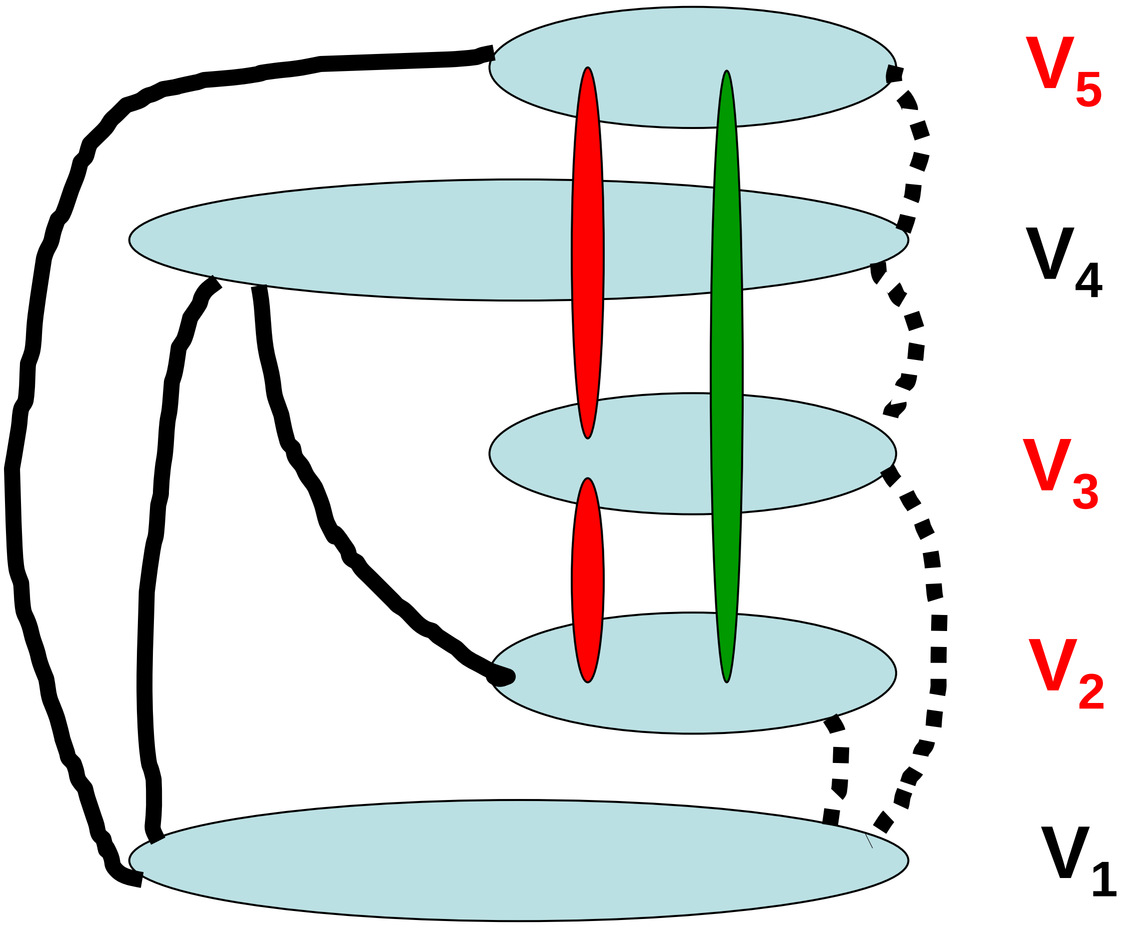
$$\varepsilon^{\Theta(\log(1/\varepsilon))} n^3$$

triangles.

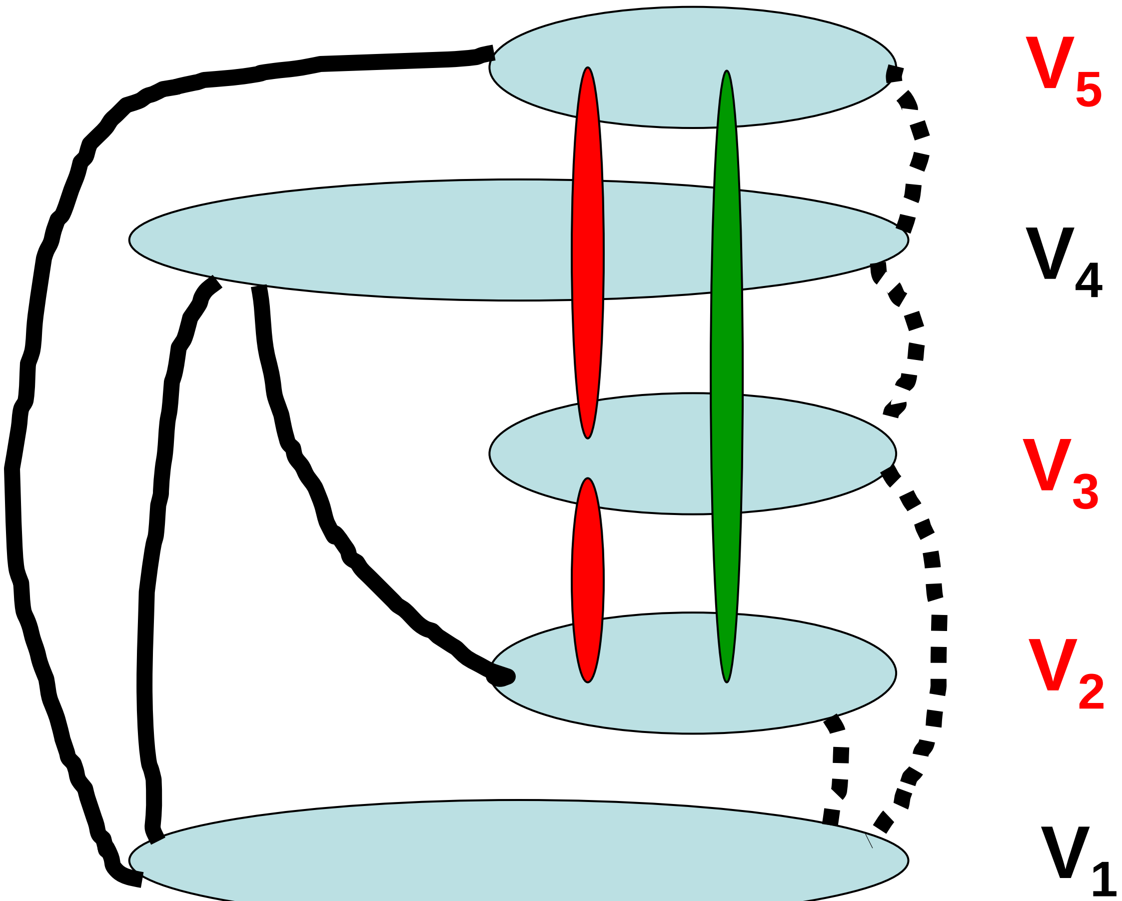
Let V_2,V_3,V_5 be the vertex classes of F, each of size n/3, add vertex classes V_1,V_4 each of size 2n and construct a graph as follows.



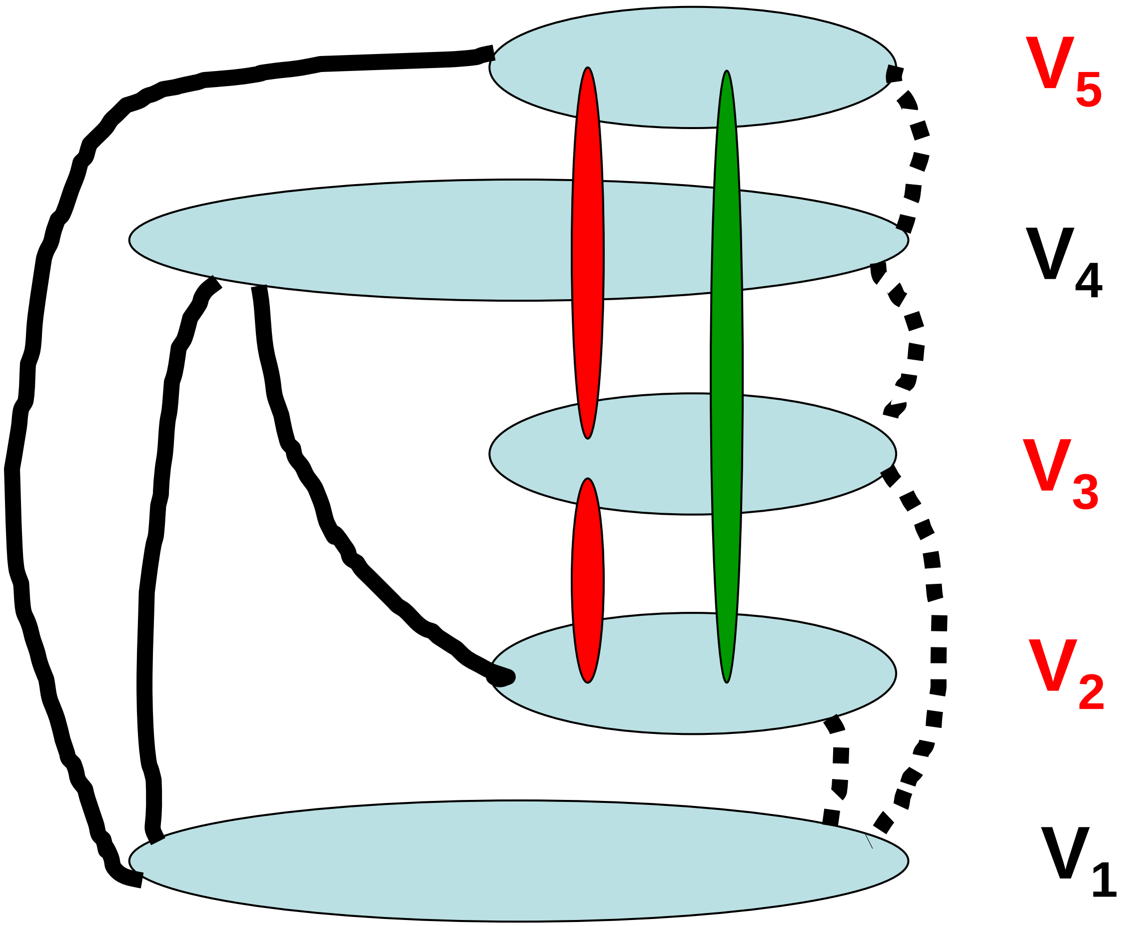
Each V_i is independent, V_1V_4, V_2V_4, V_1V_5 complete bipartite, $V_4V_5, V_3V_4, V_1V_3, V_1V_2$ no edges



V_3V_5, V_2V_3 as in F
V_2V_5 as in the bipartite complement of F



Each triangle in F with a vertex in V_1 and a vertex in V_4 give an induced C_5 . Thus G is Θ(ε)-far from induced C_5 free (and hence also from being perfect)

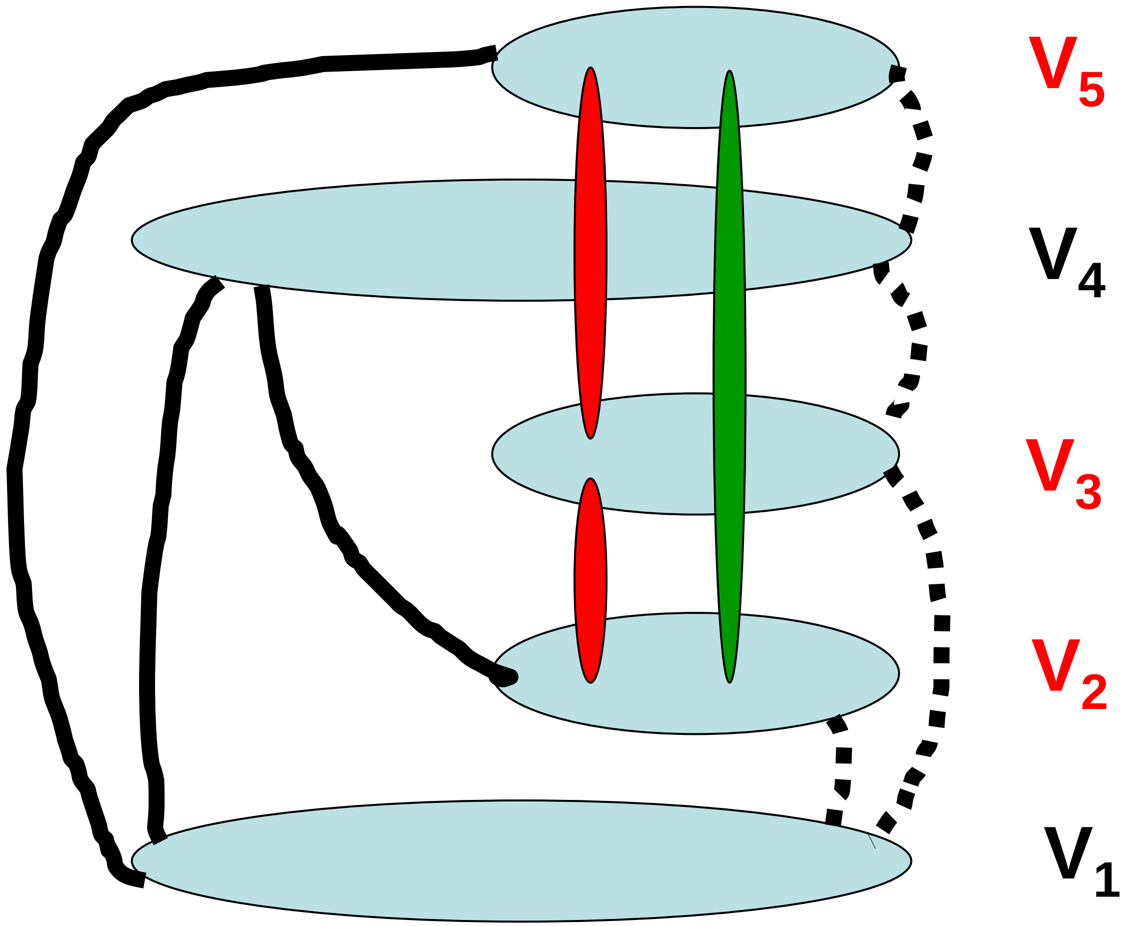


Def: A comparability graph is a graph whose vertices are the elements of a partially ordered set with two adjacent iff comparable.

It is well known that each such graph is perfect.

A graph is a comparability graph iff it has an acyclic orientation satisfying transitivity.

Orienting all edges of our graph from bottom to top, the only violations of transitivity arise from triangles in F. Thus our sample will be a comparability graph unless it contains the vertices of such a triangle.



But the number of triangles is only

$$\varepsilon^{\Theta(\log(1/\varepsilon))}\, n^3$$

and hence the probability to include one by a random set of poly(1/ε) vertices is tiny.

Hence perfectness is not easily testable, and so is the property of being a comparability graph.

Open Problems

Which (hereditary) graph properties are easily testable ?

Which properties admit witnesses of size poly(1/ε) ?

In particular, suppose that G=(V,E) is ε-far from being perfect. Must G contain an odd hole or antihole of size at most poly(1/ε) ?