Every graph is easy or hard: dichotomy theorems for graph problems

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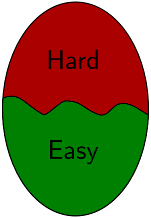
Dichotomy theorems

What is better than proving one nice result?
Proving an infinite set of nice results.

We survey results where we can precisely tell which graphs make the problem easy and which graphs make the problem hard.



Focus will be on

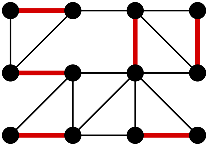
- how to formulate questions that lead to such results and
- what results of this type are known,

but less on how to prove such results.

Factor problems

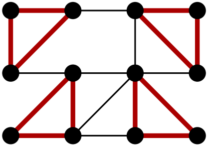
PERFECT MATCHING
Input: graph G.
Task: find $|V(G)|/2$ vertex-disjoint edges.

Polynomial-time solvable [Edmonds 1961].



TRIANGLE FACTOR
Input: graph G.
Task: find $|V(G)|/3$ vertex-disjoint triangles.

NP-complete [Karp 1975]



Factor problems

H-FACTOR
Input: graph G.
Task: find $|V(G)|/|V(H)|$ vertex-disjoint copies of H in G.

Polynomial-time solvable for $H = K_2$ and NP-hard for $H = K_3$.

Which graphs H make H-FACTOR easy and which graphs make it hard?

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Which graphs H make H-FACTOR easy and which graphs make it hard?

Theorem [Kirkpatrick and Hell 1978]

H-FACTOR is NP-hard for every connected graph H with at least 3 vertices.

Factor problems

Instead of publishing

Kirkpatrick and Hell: NP-completeness of packing cycles. 1978.
Kirkpatrick and Hell: NP-completeness of packing trees. 1979.
Kirkpatrick and Hell: NP-completeness of packing stars. 1980.
Kirkpatrick and Hell: NP-completeness of packing wheels. 1981.
Kirkpatrick and Hell: NP-completeness of packing Petersen graphs. 1982.
Kirkpatrick and Hell: NP-completeness of packing Starfish graphs. 1983.
Kirkpatrick and Hell: NP-completeness of packing Jaws. 1984.

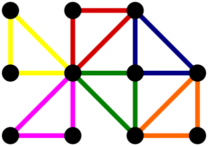
⋮

they only published

Kirkpatrick and Hell: On the Completeness of a Generalized Matching Problem. 1978

Edge-disjoint version

H-DECOMPOSITION
Input: graph G.
Task: find $|E(G)|/|E(H)|$ **edge-disjoint** copies of H in G.



- Trivial for $H = K_2$.
- Can be solved by matching for P_3 (path on 3 vertices).

Theorem [Holyer 1981]

H-DECOMPOSITION is NP-complete if H is the clique K_r or the cycle C_r for some $r \geq 3$.

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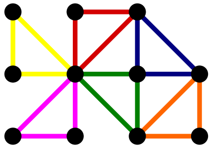

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Theorem (Holyer's Conjecture) [Dor and Tarsi 1992]

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Edge disjoint vs. vertex disjoint

It is more difficult to work with H-DECOMPOSITION than with H-FACTOR.

Edge disjoint vs. vertex disjoint

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Partition of cliques is not trivial:
Finding vertex-disjoint copies of H in a clique is trivial, but highly nontrivial for edge-disjoint copies.

Theorem [Wilson 1976]

Let m be the number of edges of H and let g be the g.c.d. of the degrees of H. The conditions $m \mid \binom{n}{2}$ and $g \mid n-1$ are obvious necessary conditions for K_n having an H-decomposition, but it is also sufficient if n is greater than some constant $n_0(H)$.

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Disconnected H is not trivial:
Problems for disconnected H can be interesting for H-DECOMPOSITION: having n edge-disjoint copies of $2 \cdot P_3$ is not the same as having $2n$ edge-disjoint copies of P_3.

H-coloring

A **homomorphism** from G to H is a mapping $f\colon V(G) \to V(H)$ such that if ab is an edge of G, then $f(a)f(b)$ is an edge of H.

H-COLORING
Input: graph G.
Task: find a homomorphism from G to H.

- If $H = K_r$, then equivalent to r-COLORING.
- G being $|V(H)|$-colorable is a necessary condition (if H has no loops).
- If H is bipartite, then the problem is equivalent to G being bipartite.

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Theorem [Hell and Nešetřil 1990]

For every simple graph H, H-COLORING is polynomial-time solvable if H is bipartite and NP-complete if H is not bipartite.

What about directed graphs?

More general homomorphism problems

Relational structures: something like edge-colored hypergraphs (edges are r-tuples of vertices).

$\textsc{Hom}(-, \mathbf{B})$
Input: a relational structure $\mathbf{A}$.
Task: find a homomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Conjecture [Feder and Vardi 1998]

For every relational structure $\mathbf{B}$, the problem $\textsc{Hom}(-, \mathbf{B})$ is either polynomial-time solvable or NP-complete.

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Theorem [Feder and Vardi 1998]

For every relational structure $\mathbf{B}$, there is a directed graph H such that $\text{Hom}(-, \mathbf{B})$ and H-coloring are polynomial-time equivalent.

Dichotomy theorems

Dichotomy theorem: classifying every member of a family of problems as easy or hard.

Why are such theorems surprising?

1. The characterization of easy/hard is a simple combinatorial property.

   So far, we have seen:

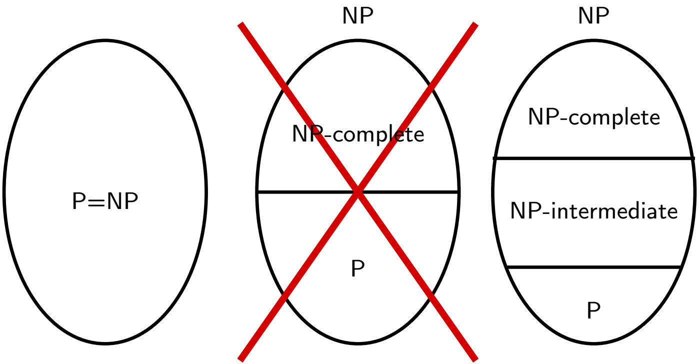
   - at least 3 vertices,
   - nonbipartite.

Dichotomy theorems

2. Every problem is either in P or NP-complete, there are no NP-intermediate problems in the family.

Theorem [Ladner 1973]

If $P \neq NP$, that there is language $L \notin P$ that is not NP-complete.



Dichotomy theorems

- Dichotomy theorems give goods research programs: easy to formulate, but can be hard to complete.
- The search for dichotomy theorems may uncover algorithmic results that no one has thought of.
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So far:
Each problem in the family was defined by fixing a graph H.

Next:
Each problem is defined by fixing a **class** of graph $\mathcal{H}$.

Hereditary deletion problems

$\mathcal{H}$-Deletion
Input: a graph G and an integer k.
Task: find a set S of k vertices such that $G - S \in \mathcal{H}$

Examples:

- $\mathcal{H}$ is the set of all graphs without edges: Vertex Cover.
- $\mathcal{H}$ is the set of all acyclic graphs: Feedback Vertex Set.

$\mathcal{H}$ is **hereditary** if it is closed under taking induced subgraphs.

Hereditary:

- planar
- chordal
- interval
- bipartite

Not hereditary:

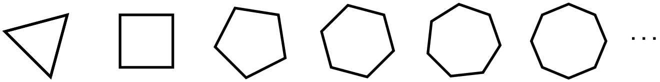
- connected
- 3-regular
- Hamiltonian
- nonbipartite

Hereditary deletion problems

Theorem [Yannakakis 1978]

For every hereditary class $\mathcal{H}$, the $\mathcal{H}$-DELETION problem is NP-complete.

Hereditary class $\mathcal{H}$ can be characterized by a (finite or infinite) list of minimal forbidden induced subgraphs.

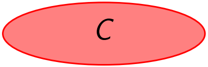


Hereditary deletion problems

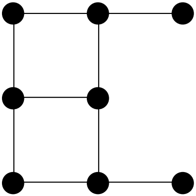
Theorem [Yannakakis 1978]

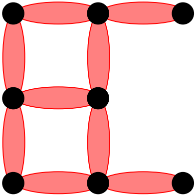
For every hereditary class $\mathcal{H}$, the $\mathcal{H}$-DELETION problem is NP-complete.

Simpler case: suppose that every minimal forbidden induced subgraph is 2-connected and let C be the smallest forbidden induced subgraph.



Reduction from VERTEX COVER:





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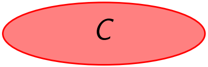

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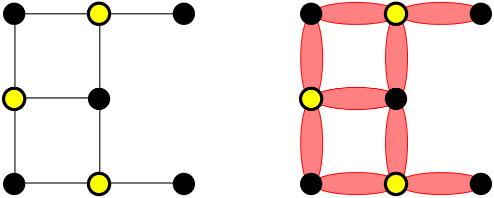

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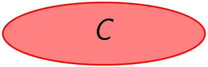

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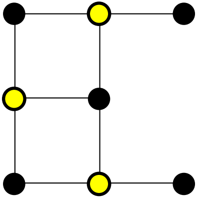
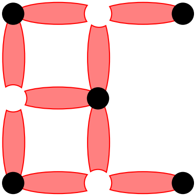
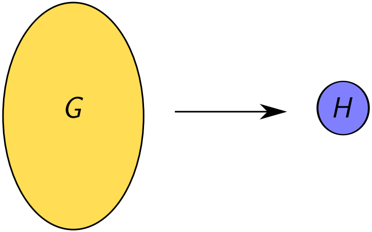




Homomorphisms seen from the other side

Recall: H-COLORING (finding a homomorphism to H) is polynomial-time solvable if H is bipartite and NP-complete otherwise.



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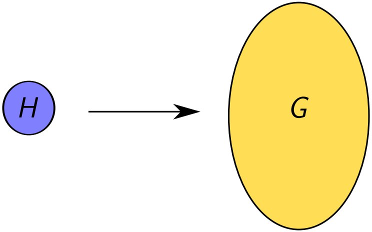

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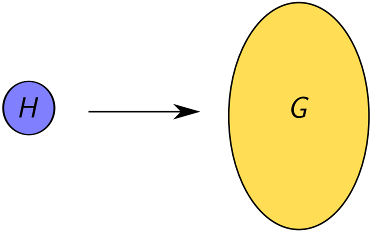

What about finding a homomorphism *from* H?

Theorem (trivial)

For every fixed H, the problem HOM$(H, -)$ (find a homomorphism from H to the given graph G) is polynomial-time solvable.

... because we can try all $|V(G)|^{|V(H)|}$ possible mappings $f: V(H) \to V(G)$.

Homomorphisms seen from the other side

Better question:

$\text{Hom}(\mathcal{H}, -)$ **Input:** a graph $H \in \mathcal{H}$ and an arbitrary graph G. **Task:** find a homomorphism from H to G.

Goal: characterize the classes $\mathcal{H}$ for which $\text{Hom}(\mathcal{H}, -)$ is polynomial-time solvable.

For example, if $\mathcal{H}$ contains only bipartite graphs, then $\text{Hom}(\mathcal{H}, -)$ is polynomial-time solvable.

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We have reasons to believe that there is no P vs. NP-complete dichotomy for $\textsc{Hom}(\mathcal{H}, -)$. Instead of NP-completeness, we will use a different tool for giving negative evidence.

Fixed-parameter tractability

More refined analysis of the running time: we express the running time as a function of input size n and a parameter k.

Definition

A problem is **fixed-parameter tractable (FPT)** parameterized by k if it can be solved in time $f(k) \cdot n^{O(1)}$ for some computable function f.

Examples of FPT problems:

- Finding a vertex cover of size k.
- Finding a feedback vertex set of size k.
- Finding a path of length k.
- Finding k vertex-disjoint triangles.
- ...

W[1]-hardness

Negative evidence similar to NP-completeness. If a problem is **W[1]-hard,** then the problem is not FPT, unless FPT = W[1].

Some W[1]-hard problems:

- Finding a clique/independent set of size k.
- Finding a dominating set of size k.
- Finding k pairwise disjoint sets.
- ...

For these problems, the exponent of n has to depend on k (the running time is typically $n^{O(k)}$).

Counting Homomorphisms

... back to homomorphisms.

$\#\textsc{Hom}(\mathcal{H}, -)$
Input: a graph $H \in \mathcal{H}$ and an arbitrary graph G.
Task: count the number of homomorphisms from $H \to G$.

We parameterize by $k = |V(H)|$, i.e., our goal is an $f(|V(H)|) \cdot n^{O(1)}$ time algorithm.

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Theorem [Dalmau and Jonsson 2004]

Assuming FPT $\neq$ W[1], for every recursively enumerable class $\mathcal{H}$ of graphs, the following are equivalent:

1. $\#\textsc{Hom}(\mathcal{H}, -)$ is polynomial-time solvable.
2. $\#\textsc{Hom}(\mathcal{H}, -)$ is FPT parameterized by $|V(H)|$.
3. $\mathcal{H}$ has bounded treewidth.

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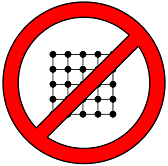
Excluded Grid Theorem [Robertson and Seymour]

There is a function f such that every graph with treewidth $f(k)$ contains a $k \times k$ grid minor.



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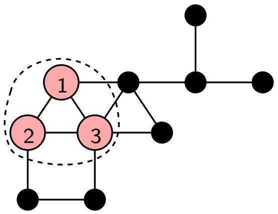
Steps of the proof:

- Show that the problem is polynomial-time solvable for bounded treewidth.
- Show that the problem is W[1]-hard if $\mathcal{H}$ is the class of grids.
- Use the Excluded Grid Theorem to show that this implies W[1]-hardness for every class with unbounded treewidth.

Decision version

$\textsc{Hom}(\mathcal{H}, -)$ **Input:** a graph $H \in \mathcal{H}$ and an arbitrary graph G. **Task:** find a homomorphism from H to G.

Core of H: smallest subgraph H^* of H such that there is a homomorphism $H \to H^*$ (known to be unique up to isomorphism).



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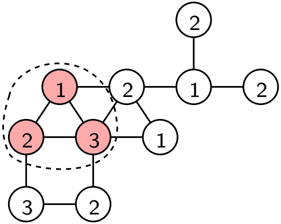

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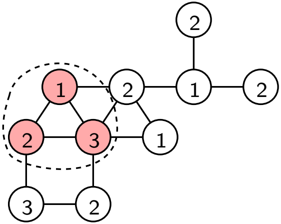

Observation

If H^* is the core of H, then there is a homomorphism $H^* \to G$ if and only if there is a homomorphism $H \to G$.

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Theorem [Grohe 2003]

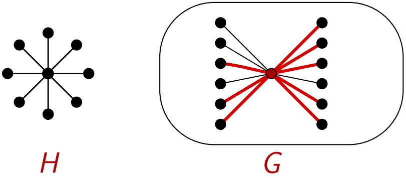
Assuming $\text{FPT} \neq \text{W[1]}$, for every recursively enumerable class $\mathcal{H}$ of graphs, the following are equivalent:

1. $\textsc{Hom}(\mathcal{H}, -)$ is polynomial-time solvable.
2. $\textsc{Hom}(\mathcal{H}, -)$ is FPT parameterized by $|V(H)|$.
3. there is a constant $c \geq 1$ such that the core of every graph in $\mathcal{H}$ has treewidth at most c.

Counting Subgraphs

$\#\textsc{Sub}(\mathcal{H})$
Input: a graph $H \in \mathcal{H}$ and an arbitrary graph G.
Task: calculate the number of copies of H in G.

If $\mathcal{H}$ is the class of all stars, then $\#\textsc{Sub}(\mathcal{H})$ is easy: for each placement of the center of the star, calculate the number of possible different assignments of the leaves.

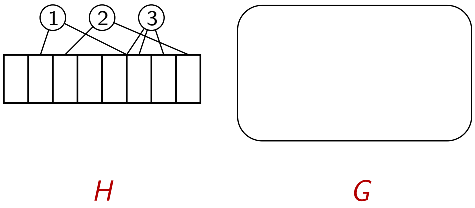


Counting Subgraphs

$\#\textsc{Sub}(\mathcal{H})$
Input: a graph $H \in \mathcal{H}$ and an arbitrary graph G.
Task: calculate the number of copies of H in G.

Theorem

If every graph in $\mathcal{H}$ has vertex cover number at most c, then $\#\textsc{Sub}(\mathcal{H})$ is polynomial-time solvable.



Running time is $n^{2^{O(c)}}$, better algorithms known [Vassilevska Williams and Williams], [Kowaluk, Lingas, and Lundell].

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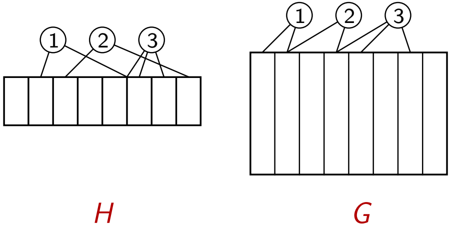

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Counting subgraphs

Who are the bad guys now?

Theorem [Flum and Grohe 2002]

If $\mathcal{H}$ is the set of all paths, then $\#\textsc{Sub}(\mathcal{H})$ is #W[1]-hard.

Theorem [Curticapean 2013]

If $\mathcal{H}$ is the set of all matchings, then $\#\textsc{Sub}(\mathcal{H})$ is #W[1]-hard.

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Dichotomy theorem:

Theorem [Curticapean and M. 2014]

Let $\mathcal{H}$ be a recursively enumerable class of graphs. If $\mathcal{H}$ has unbounded vertex cover number, then $\#\text{Sub}(\mathcal{H})$ is #W[1]-hard.

($\nu(G) \le \tau(G) \le 2\nu(G)$, hence "unbounded vertex cover number" and "unbounded matching number" are the same.)

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Dichotomy theorem:

Theorem [Curticapean and M. 2014]

Let $\mathcal{H}$ be a recursively enumerable class of graphs. If $\mathcal{H}$ has unbounded vertex cover number, then $\#\textsc{Sub}(\mathcal{H})$ is #W[1]-hard.

($\nu(G) \le \tau(G) \le 2\nu(G)$, hence "unbounded vertex cover number" and "unbounded matching number" are the same.)

There is a simple proof if $\mathcal{H}$ is hereditary, but the general case is more difficult.

Counting subgraphs

Observation

At least one of the following holds for every hereditary class $\mathcal{H}$ with unbounded vertex cover number:

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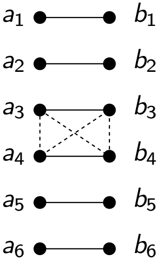
- For every $i < j$, there are 2^4 possibilities for the 4 edges between $\{a_i, b_i\}$ and $\{a_j, b_j\}$.
- If there is a large matching, then there is a large matching that is homogeneous with respect to these 16 possibilities.



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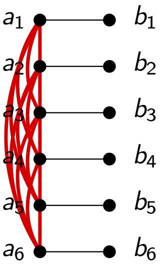

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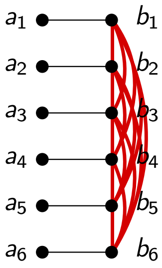

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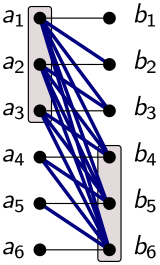

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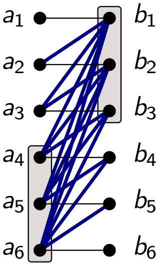

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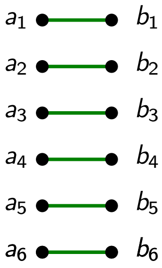
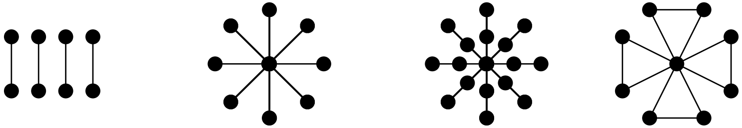


Finding subgraphs

> $\textsc{Sub}(\mathcal{H})$
> **Input:** a graph $H \in \mathcal{H}$ and an arbitrary graph G.
> **Task:** decide if H is a subgraph of G.

Some classes for which $\textsc{Sub}(\mathcal{H})$ is polynomial-time solvable:

- $\mathcal{H}$ is the class of all matchings
- $\mathcal{H}$ is the class of all stars
- $\mathcal{H}$ is the class of all stars, each edge subdivided once
- $\mathcal{H}$ is the class of all windmills

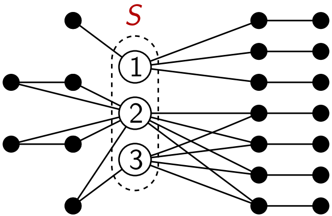


matching star subdivided star windmill

Finding subgraphs

Definition

Class $\mathcal{H}$ is **matching splittable** if there is a constant c such that every $H \in \mathcal{H}$ has a set S of at most c vertices such that every component of $H - S$ has size at most 2.



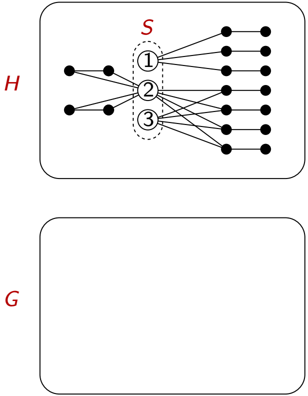
Theorem [Jansen and M. 2014]

Let $\mathcal{H}$ be a hereditary class of graphs. If $\mathcal{H}$ is matching splittable, then $\textsc{Sub}(\mathcal{H})$ is randomized polynomial-time solvable and NP-hard otherwise.

Finding subgraphs (algorithm)

Theorem [Jansen and M. 2014]

If hereditary class $\mathcal{H}$ is matching splittable, then $\textsc{Sub}(\mathcal{H})$ is randomized polynomial-time solvable.



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If hereditary class $\mathcal{H}$ is matching splittable, then $\textsc{Sub}(\mathcal{H})$ is randomized polynomial-time solvable.

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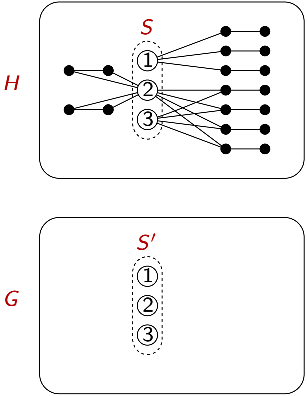

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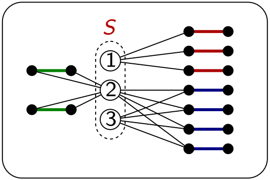
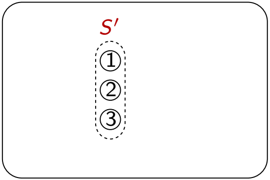
H



G



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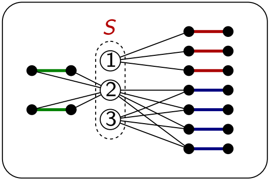
- Guess the image S' of S in G.
- Classify the edges of $H - S$ according to their neighborhoods in S (at most 2^{2c} colors).
- Classify the edges of $G - S'$ according to which edge of $H - S$ can be mapped into it (use parallel edges if needed).
- Task is to find a matching in $G - S'$ with a certain number of edges of each color.

H



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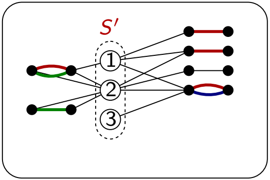

Finding subgraphs (algorithm)

Theorem [Mulmuley, Vazirani, Vazirani 1987]

There is a randomized polynomial-time algorithm that, given a graph G with red and blue edges and integer k, decides if there is a perfect matching with exactly k red edges.

More generally:

Theorem

Given a graph G with edges colored with c colors and c integers k_1, $\ldots$, k_c, we can decide in randomized time $n^{O(c)}$ if there is a matching with exactly k_i edges of color i.

This is precisely what we need to complete the algorithm for $\textsc{Sub}(\mathcal{H})$ for matching splittable $\mathcal{H}$.

Finding subgraphs (hardness proof)

Lemma

Let $\mathcal{H}$ be a hereditary class of graphs that is not matching splittable. Then at least one of the following is true.

- $\mathcal{H}$ contains every clique.
- $\mathcal{H}$ contains every biclique.
- For every $n \geq 1$, $\mathcal{H}$ contains $n \cdot K_3$.
- For every $n \geq 1$, $\mathcal{H}$ contains $n \cdot P_3$ (where P_3 is the path on 3 vertices).

In each case, $\textsc{Sub}(\mathcal{H})$ is NP-hard (recall that P_3-FACTOR and K_3-FACTOR are NP-hard).

Finding subgraphs (hardness proof)

Recall: Class $\mathcal{H}$ is **matching splittable** if there is a constant c such that every $H \in \mathcal{H}$ has a set S of at most c vertices such that every component of $H - S$ has size at most 2.

Equivalently: in every $H \in \mathcal{H}$, we can cover every 3-vertex connected set (i.e., every K_3 and P_3) by c vertices.

Observation: either

- there are r vertex disjoint K_3, or
- there are r vertex disjoint P_3, or
- we can cover every K_3 and every P_3 by $6r$ vertices.

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- Consider many vertex-disjoint P_3's.
- For every $i < j$, there are 2^9 possibilities between $\{a_i, b_i, c_i\}$ and $\{a_j, b_j, c_j\}$.
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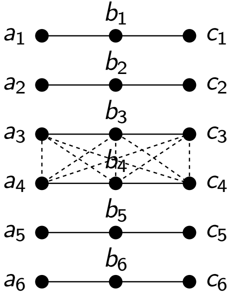

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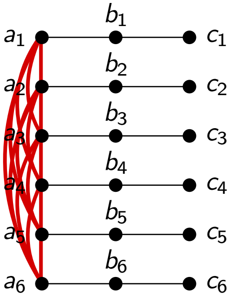

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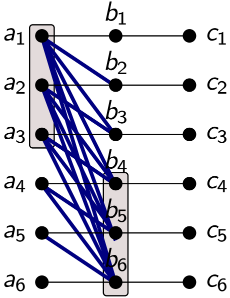

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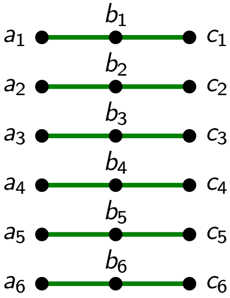

Summary

Dichotomy results:

- P vs. NP-hard or FPT vs. W[1]-hard.
- For a fixed graph H or (hereditary) class $\mathcal{H}$.

Considered problems:

- H-FACTOR
- H-DECOMPOSITION
- H-COLORING
- $\mathcal{H}$-DELETION
- HOM$(\mathcal{H}, -)$
- #HOM$(\mathcal{H}, -)$
- #SUB$(\mathcal{H})$
- SUB$(\mathcal{H})$

Conclusions

- For numerous problems, we can prove that every fixed graph (or graph class) is either easy or hard.
- Good research programs: easy to formulate, hard to solve, but not completely impossible.
- Possible outcomes:
  - Everything is hard, except some trivial cases.
  - Everything is hard, except the famous known nontrivial positive cases.
  - Some unexpected easy cases are found.
- Requires attacking the problem both from the algorithmic and the complexity side.