Three-edge-colouring cubic doublecross graphs

Paul Seymour, Princeton University
(with Katie Edwards, Dan Sanders, Robin Thomas.)

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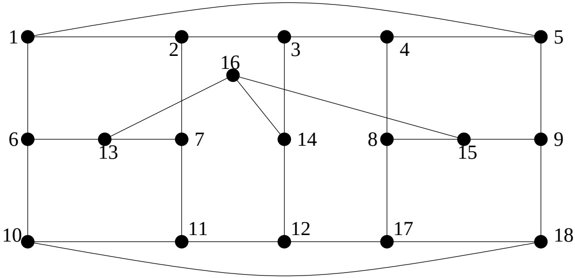
Every bridgeless planar cubic graph is three-edge-colourable.

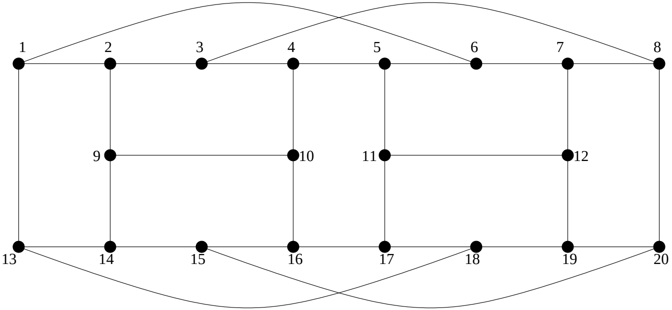
Tutte's conjecture, 1966

Every bridgeless cubic graph not containing Petersen as a minor is three-edge-colourable.

G is apex if $G \setminus v$ is planar for some vertex v.



G is doublecross if *G* can be drawn in the plane with only two crossings, both on the outside.



G is theta-connected if

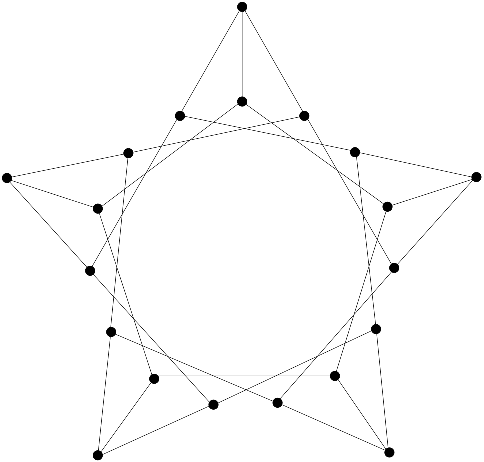
- G is cubic and $|V(G)| \geq 8$;
- for every partition (X, Y) of $V(G)$ with $|X|, |Y| \geq 3$ there are at least five edges between X and Y;
- for every partition (X, Y) of $V(G)$ with $|X|, |Y| \geq 7$ there are at least six edges between X and Y.

Theorem (Robertson, S., Thomas, 1997)

Any minimal counterexample to Tutte's conjecture is either theta-connected or apex.

Theorem (Robertson, S., Thomas, 1995)

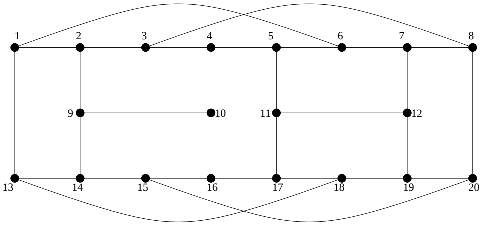
Every theta-connected graph not containing Petersen is either apex or doublecross, except Starfish.



Theorem

Let G be theta-connected, and not contain Petersen. If G

- *contains Starfish then G is Starfish*
- *contains Jaws then G is doublecross*
- *contains neither of Jaws and Starfish then G is apex.*



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Every minimal counterexample to Tutte's conjecture is theta-connected, and either apex or doublecross.

To prove Tutte's conjecture in general, it is enough to prove that

- every bridgeless apex cubic graph is three-edge-colourable (proved by Sanders and Thomas $\sim$1997)
- every bridgeless non-apex doublecross cubic graph is three-edge-colourable (proved by Edwards, Sanders, S., Thomas 2014).

Theorem

(Assuming 4CT) every bridgeless cubic graph with crossing number one is three-edge-colourable.

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Let X be a connected planar graph with all vertices of degree three, and with half-edges going into the infinite region. Let Ω be the half-edges in order, and let $\mathcal{C}(\Omega, X)$ be the set of all three-colourings of Ω that can be extended to three-edge-colourings of X.

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Theorem

$\mathcal{C}(\Omega, X)$ is planar-consistent.

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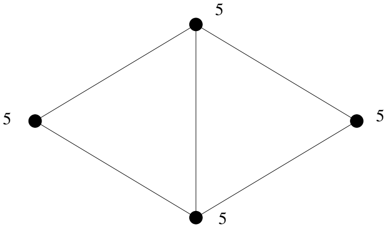

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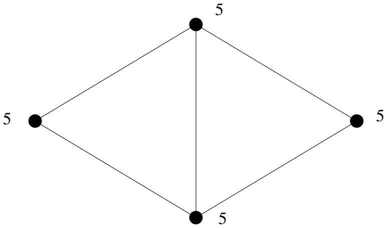

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Birkhoff's diamond is D-reducible.

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Now there are thousands of configurations that are known to be D- or C-reducible. We used 633 of them.

Unavoidability

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Enough to show:

Theorem

One of the 633 appears in every internally 6-connected planar triangulation.

Theorem

If T is an internally 6-connected triangulation, there is a function $\phi(u, v)$ for all adjacent u, v, satisfying:

- $\phi(u, v) = -\phi(v, u)$
- *if $\phi(u, v) > 5$ then one of the 633 configurations is present and contains u*
- *if $10(6 - d(u)) - \sum_v \phi(u, v) > 0$ (where $d(u)$ is the degree of u and the sum is over all vertices v adjacent to u) then one of the 633 configurations is present, and either u or some neighbour of u is contained in it.*

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- Step 1: Change "planar-consistent" to "XX-consistent". (Use doublecross pairings instead of planar pairings.)
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Not all the 633 are XX-consistent. But we found a list of 756 that works.

- All 756 configurations are XXD- or XXC-reducible
- The discharging theorem still works (all three parts) with the same function $\phi(u, v)$.

Back to the cubic graph G: let g_1, g_2, g_3, g_4 be the crossing edges. Choose its drawing so that $g_1, \ldots, g_4$ are in the infinite region R_∞ of $G \setminus \{g_1, \ldots, g_4\}$. Let Z be the cycle bounding R_∞.

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Theorem

If $|E(Z)| \geq 21$ then one of the 756 configurations appears (in its cubic form) in $G \setminus \{g_1, \ldots, g_4\}$, with all its finite regions disjoint from R_∞.

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- The sum of $10(6 - d(u))$, summed over all $u \in V(T) \setminus V(C)$, equals $10(k + 6 - 2|V(C)|)$, where k is the number of edges of T between $V(C)$ and $V(T) \setminus V(C)$.

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- So at least $5k + 60 - 20|V(C)|)$ remains on the vertices in $V(T) \setminus V(C)$. But $|V(C)| = 8$ and $k \geq 21$, so some vertex in $V(T) \setminus V(C)$ has positive charge.