Orientations and colouring of graphs

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There are several known results that reveal the connection between the colourings of a graph and its orientations. However it seems that there are only the tip of the iceberg. In these notes, we present some of the most important links between these two notions. On the way, we will present some nice proof techniques.

1 Definitions and basic concepts

All graphs considered is these notes are simple, that is they have no loops nor multiple edges. We rely on [15] for classical notation and concepts.

In the remaining of this section, we recall some useful definitions and well-known theorems.

1.1 Graphs and digraphs

For a graph G, we denote by $v(G)$ its number of vertices and by $e(G)$ its number of edges. For a digraph D, we denote by $v(G)$ its number of vertices and by $a(G)$ its number of arcs.

If G is a (directed) graph, then the (directed) subgraph induced by a set S of vertices is denoted $G\langle S\rangle$.

A graph or digraph is *empty* if it has no edges or no arcs, respectively.

If (x,y) is an arc, we write $x \rightarrow y$. More generally, if $x \rightarrow y$ for all $x \in X$, $y \in Y$, then we write $X \rightarrow Y$. We abbreviate $\{x\} \rightarrow Y$ in $x \rightarrow Y$.

A *source* is a vertex of indegree 0, and a *sink* a vertex of outdegree 0.

A digraph is *eulerian* if $d^+(v) = d^-(v)$ for every vertex v.

The *average degree* of a graph G is $\mathrm{Ad}(G) := \frac{1}{v(G)} \sum_{v \in V(G)} d(v) = \frac{2e(G)}{v(G)}$. The *maximum average degree* of G is $\mathrm{Mad}(G) := \max\{\mathrm{Ad}(H) \mid H \text{ is a subgraph of } G\}$.

We denote by $UG(D)$ the underlying (multi)graph of D, that is, the (multi)graph we obtain by replacing each arc by an edge. The digraph D is *connected* (resp. *k-connected*) if $UG(D)$ is a connected (resp. k-connected) graph. It is *strongly connected*, or *strong*, if for any two vertices u,v, there is a directed (u,v)-path in D.

A *handle decomposition* of D is a sequence $H_1, \dots, H_r$ such that:

i) H_1 is a directed cycle of D.

ii) For every $i = 2, \dots, r$, H_i is a handle, that is, a directed path of D (with possibly the same endvertices) starting and ending in $V(H_1 \cup \dots \cup H_{i-1})$ but with no inner vertex in this set.

iii) $D = H_1 \cup \dots \cup H_r$.

An H_i which is an arc is a *trivial handle*. It is well-known that every strong digraph admits a handle decomposition and that the number r of handles is invariant for all handle decompositions of D (indeed, $r = a(D) - v(D) + 1$).

1.2 Orientations

An *orientation* of a graph G is a digraph obtained from G by replacing each edge by just one of the two possible arcs with the same ends. In other words, an orientation of G is a digraph D whose underlying graph is G. In particular, an orientation of a (simple) graph has no opposite arcs, (so no directed 2-cycles). An orientation of a graph is called an *oriented graph*. Similarly, an orientation of a path, cycle, or tree, is a called an *oriented path*, *oriented cycle*, or *oriented tree*, respectively.

An *antidirected graph* is an oriented graph in which every vertex is either a source or a sink. Notice that an antidirected graph is thus necessarily bipartite. A *rooted tree* $T(x)$ is a tree T with a specified vertex x, called the *root* of T. An orientation of a rooted tree in which every vertex but the root has indegree 1 (resp. outdegree 1) is called an *out-arborescence* (resp. *in-arborescence*). An *arborescence* is either an out-arborescence or an in-arborescence. An *out-forest* is a forest of out-arborescences, that is the disjoint union of out-arborescences.

An orientation of a complete graph is a *tournament*. An acyclic tournament is *transitive*.

The following useful lemma was first proved by Hakimi [48]. It also appears independently in several papers [88, 5, 6]. The proof presented here follows [88] and [6].

Lemma 1.1. *A graph G has an orientation D with maximum outdegree at most Δ^+ if and only if* $\mathrm{Mad}(G)/2 \leqslant \Delta^+$.

Proof. Suppose first that G has such an orientation D. Then for any subgraph H of G

$$e(H) = \sum_{v \in V(H)} d_H^+(v) \leqslant \sum_{v \in V(H)} d_D^+(v) \leqslant \Delta^+ \cdot v(H)$$

and hence $e(H)/v(H) \leqslant \Delta^+$. Thus $\mathrm{Mad}(G)/2 \leqslant \Delta^+$.

Suppose now that $\mathrm{Mad}(G)/2 \leqslant \Delta^+$. Let F be the bipartite graph with bipartition (A,B), where $A = E(G)$ and B is a union of Δ^+ disjoint copies $V_1, \dots, V_{\Delta^+}$ of V. Each edge $uv \in E(G)$ is joined in F to the Δ^+ copies of u and the Δ^+ copies of v in B. We claim that F contains a matching of size $|A| = e(G)$. Indeed, if $E' \subset E$ is a set of edges of a subgraph H of G whose vertices are all endpoints of members of E', then, in F, E' has $\Delta^+ \cdot v(H)$ neighbours. By the definition of $\mathrm{Mad}(G)$, $|E'|/v(H) \leqslant \mathrm{Mad}(G)/2 \leqslant \Delta^+$. Hence $\Delta^+ \cdot v(H) \geqslant |E'|$. Therefore, by Hall's Theorem, the desired matching exists. We can now orient each edge of G from the vertex to which it is matched. This gives an orientation D of G with maximum degree at most Δ^+. $\square$

1.3 Vertex colouring and list colouring

A *(vertex) colouring* of a graph G is a mapping $c : V(G) \to S$. The elements of S are called *colours*; the vertices of one colour form a *colour class*. If $|S| = k$, we say that c is a *k-colouring* (often we use $S = \{1, \ldots, k\}$). A colouring is *proper* if adjacent vertices have different colours. A graph is *k-colourable* if it has a proper k-colouring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colourable. Obviously, $\chi(G)$ exists as assigning distinct colours to vertices yields a proper $v(G)$-colouring. An *optimal colouring* of G is a $\chi(G)$-colouring. A graph G is *k-chromatic* if $\chi(G) = k$.

In a proper colouring, each colour class is a stable set. Hence a *k-colouring* may also be seen as a partition of the vertex set of G into k disjoint *stable sets* $S_i = \{v \mid c(v) = i\}$ for $1 \leqslant i \leqslant k$. Therefore k-colourable graphs are also called *k-partite graphs*. Moreover, 2-colourable graphs are very often called *bipartite*. A bipartite graph G with colour classes X and Y is denoted $((X,Y),E(G))$.

Trivially $\chi(G) \geqslant \omega(G)$, where $\omega(G)$ denotes the maximum size of a clique in G, because in any proper colouring the vertices of a clique receive distinct colours. On the other hand, there are graphs with clique number 2 and arbitrarily large chromatic number as shown by a number of mathematicians including Descartes (alias Tutte) [24], Kelly and Kelly [59], Zykov [96] or Mycielski [70].

The *girth* of a graph is the smallest length of a cycle (or $+\infty$ if the graph is acyclic). A graph with clique number 2 has girth at least 4. Erdös [31] applied the probabilistic method to demonstrate the existence of graphs with arbitrarily high girth and chromatic number.

Theorem 1.2 (Erdös [31])**.** *For any two positive integers g and k , there exists a graph G with girth larger than g and chromatic number larger than k.*

Most upper bounds on the chromatic number come from algorithms that produce proper colourings. The most widespread one is the greedy algorithm. Given a linear order of $\sigma = v_1, \ldots, v_n$, the greedy colouring proceeds as follows. For $i = 1$ to n, it assigns to the vertex v_i the smallest colour (colours are positive integers in this case) which is not already used on one of its neighbours.

In fact, there are two variants of the greedy algorithm. In the ”one-pass” variant, which we just described, we run through the vertices in order and always assign the smallest available colour. In the ”many-passes” variant, we run through the vertices assigning colour 1 whenever possible, then repeat with colour 2 and so on. However, for proper colourings, both methods yield exactly the same colouring.

For a linear order σ, let $\gamma_\sigma(G)$ be the number of colours used by the greedy algorithm with respect to σ. Clearly, $\chi(G) \leqslant \gamma_\sigma(G)$ for all linear order σ. In fact, $\chi(G)$ is the minimum of $\gamma_\sigma(G)$ over all linear orders σ.

For every vertex v_i, let $d_\sigma(v_i)$ be the number of neighbours of v_i in $\{v_1, \ldots, v_{i-1}\}$, and set $d_\sigma(G) = \max\{d_\sigma(v) \mid v \in V(G)\}$. Clearly, $\gamma_\sigma(G) \leqslant d_\sigma(G) + 1$ since at most $d_\sigma(v_i)$ colours are forbidden for v_i when the greedy algorithm colours it.

The *degeneracy* of G, denoted by $\delta^*(G)$, is the minimum of $d_\sigma(G)$ over all linear orders σ. It is folklore that the degeneracy is the minimum integer k such that each of its subgraphs has a vertex of degree at most k. In symbols, $\delta^*(G) = \max\{\delta(H) \mid H \text{ subgraph of } G\}$. Trivially, $\delta(G) \leqslant \delta^*(G) \leqslant \Delta(G)$. We have

$$\chi(G) \leqslant \delta^*(G) + 1 \leqslant \Delta(G) + 1.$$

There are graphs G for which $\chi(G) = \Delta(G) + 1$: for the complete graph on n vertices, we have $\chi(K_n) = n = \Delta(K_n) + 1$ and for an odd cycle C, we have $\chi(C) = 3 = \Delta(C) + 1$. However, Brooks'Theorem says that these two examples are essentially the only ones.

Theorem 1.3 (Brooks [16]). BROOKS' THEOREM
Let G be a connected graph. Then $\chi(G) \leqslant \Delta(G)$ unless G is either a complete graph or an odd cycle.

The graph G is *k-critical* if $\chi(G) = k$ and $\chi(G') < k$ for every proper subgraph G' of G. A *critical* graph is a graph that is k-critical for some integer $k \geqslant 1$.

The only 1-critical graph is K_1, the only 2-critical graph is K_2, and the only 3-critical graphs are the odd cycles.

If v is a vertex of a k-critical graph G, then $G - v$ is $(k-1)$-colourable. But no proper $(k-1)$-colouring of $G - v$ can be extended into a proper $(k-1)$-colouring of G. Hence, all the $k-1$ colours must appear on the neighbourhood of v, and so $d_G(v) \geqslant k-1$. We just proved the following.

Proposition 1.4. *If G is k-critical for some integer $k \geqslant 1$, then $\delta(G) \geqslant k-1$.*

List colouring is a generalization of vertex colouring in which the set of colours available at each vertex is restricted. This model was introduced independently by Vizing [94] and Erdős-Rubin-Taylor [29].

A *list-assignment* of a graph G is an application L which assigns to each vertex $v \in V(G)$ a prescribed list of colours $L(v)$. A list-assignment is a *k-list-assignment* if each list is of size at least k. An *L-colouring* of G is a colouring c such that $c(v) \in L(v)$ for all $v \in V(G)$. A graph G is *L-colourable* if there exists a proper L-colouring of G. It is *k-choosable* if it is L-colourable for every k-list-assignment L. More generally, for $f : V(G) \to \mathbb{N}$, an *f-list-assignment* is a list-assignment L such that $|L(v)| \geqslant f(v)$ for all vertex $v \in V(G)$. A graph G is *f-choosable* if it is L-colourable for every f-list-assignment L.

The *choice number* or *list chromatic number* of G, denoted by $\mathrm{ch}(G)$, is the least k such that G is k-choosable. Since the lists could be identical, $\chi(G) \leqslant \mathrm{ch}(G)$.

The greedy algorithm may be modified to search for a proper L-colouring. For $i = 1$ to n, it assigns to the vertex v_i the smallest colour in $L(v_i)$ which is not already used on one of its neighbours. If all colours of $L(v_i)$ are used, then the greedy algorithm *fails* and stops. If the greedy algorithm does not fail, then it *succeeds*. In this case, it returns a proper L-colouring of G. If $|L(v_i)| \geqslant d_\sigma(v_i) + 1$, for $1 \leqslant i \leqslant n$, then the greedy algorithm succeeds. Therefore $\mathrm{ch}(G) \leqslant \delta^*(G) + 1$ colours.

Hence, we have the following sequence of inequalities.

$$\chi(G) \leqslant \mathrm{ch}(G) \leqslant \delta^*(G)+1 \leqslant \Delta(G)+1. \tag{1}$$

All these inequalities are tight, because if K is a complete graph $\chi(K) = \Delta(K)+1$. One the other hand, each of them can be very loose. For example, it is not possible to pace an upper bound on $\mathrm{ch}(G)$ in terms of $\chi(G)$ because there are bipartite graphs with arbitrarily large choice number.

Proposition 1.5 (Erdős, Rubin and Taylor [29]). *If $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not k-choosable.*

Proof. Let (A,B) be the bipartition of $K_{m,m}$. Let L be a list assignment such that for every k-subset I of $\{1,2,\dots,2k-1\}$ there exists a vertex $a_I \in A$ and a vertex $b_I \in B$ such that $L(a_I) = L(b_I) = I$. Consider an L-colouring of $K_{m,m}$. Let S_A be the set of colours used on A and S_B be the set of colours used on B. Then $|S_A| \geqslant k$ otherwise a vertex a_I with $I \subset \{1,2,\dots,2k-1\} \setminus S_A$ would be assigned no colour. Similarly, $|S_B| \geqslant k$. Hence $S_A \cap S_B \neq \varnothing$, so the colouring is not proper. □

Brooks' Theorem may be extended to list colouring.

Theorem 1.6. BROOKS' THEOREM (LIST VERSION)
Let G be a connected graph. Then $\mathrm{ch}(G) \leqslant \Delta(G)$ unless G is either a complete graph or an odd cycle.

A (proper) *vertex colouring* of a digraph D is simply a vertex colouring of its underlying graph G, and its *chromatic number* $\chi(D)$ is defined to be the chromatic number $\chi(G)$ of G.

1.4 Edge colouring

An *edge-colouring* of G is a mapping $f: E(G) \to S$. The element of S are *colours*; the edges of one colour form a *colour class*. If $|S| = k$, then f is a *k-edge-colouring*. An edge-colouring is *proper* if incident edges have different colours; that is, if each colour class is a matching. A graph is *k-edge-colourable* if it has a proper k-edge-colouring. The *chromatic index* or *edge-chromatic number* $\chi'(G)$ of a graph G is the least k such that G is k-edge-colourable.

Edge-colouring may be seen as a vertex colouring of a special class of graphs, namely the line graphs. The *line graph* $L(G)$ of a graph G is the graph with vertex set $E(G)$, such that $ef \in E(L(G))$ whenever e and f share an endvertex in G. Then, $\chi'(G) = \chi(L(G))$.

Since edges sharing an endvertex need different colours, $\chi'(G) \geqslant \Delta(G)$. On the other hand, as an edge is incident to at most $2\Delta(G)-2$ other edges ($\Delta - 1$ at each endvertex), colouring the edges greedily uses at most $2\Delta(G)-1$ colours. However, one does not need that many colours, as it was shown independently by Vizing [93] and Gupta [43].

Theorem 1.7 (Vizing [93], Gupta [43]). *If G is a graph, then $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$.*

A graph is said to be *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* if $\chi'(G) = \Delta(G)+1$. Holyer [58] showed that determining whether a graph is Class 1 or Class 2 is NP-complete.

However there are classes of graphs for which we know if they are Class 1 or Class 2. König [60] proved that every bipartite graph is Class 1.

Theorem 1.8 (König [60])**.** *Let G be a bipartite graph. Then $\chi'(G) = \Delta(G)$.*

Analogously to list colouring, one can define list edge-colouring. In this variant, lists of colours are assigned to the edges and we search for proper edge-colouring such that every edge is coloured with a colour of its list. The *list chromatic index* $\mathrm{ch}'(G)$ of a graph G is the least integer k such that whenever every edge of G is assigned a list of k colours, G admits a proper edge-colouring f such that $f(e) \in L(e)$ for every edge e. Equivalently, $\mathrm{ch}'(G) = \mathrm{ch}(L(G))$, where $L(G)$ is the line graph of G.

Analogously as for χ', the greedy algorithm yields that $\mathrm{ch}'(G) \leqslant 2\Delta(G) - 1$. In particular, $\mathrm{ch}'(G)$ is bounded in terms of $\chi'(G)$. It was suggested independently by many researchers — including Vizing, Gupta, Albertson, Collins, and Tucker — and it first appeared in print in a work of Bollobás and Harris [13], that the list chromatic index equals the chromatic index.

Conjecture 1.9. LIST COLOURING CONJECTURE
The chromatic index is equal to the list chromatic index, that is $\chi' = \mathrm{ch}'$.

2 Using orientations for colouring

Each linear order $\sigma = v_1, \cdots, v_n$ of the vertices of a graph G induces an acyclic orientation D of G in which v_i dominates v_j if and only if $i > j$. Observe that $d_D^+(v_i) = d_\sigma(v_i)$, $1 \leqslant i \leqslant n$. An acyclic orientation D of G may be induced by many different linear orders. However, applying the greedy algorithm according to any of these linear orders always results in the same colouring of G. This colouring c_D, called the *greedy colouring with respect to D*, is the one having the property that every vertex of D has the smallest colour not appearing on its outneighbourhood. Let us denote by $\gamma(D)$ the number of colours used by c_D.

Proposition 2.1. *Let D be an acyclic orientation of a graph G. Then*

$$\chi(G) \leqslant \gamma(D) \leqslant \Delta^+(D) + 1 \leqslant \Delta(G) + 1.$$

To be sure to use few colours when greedily colouring with respect to an acyclic orientation, a natural idea is to consider an orientation with small maximum outdegree. But the minimum of $\Delta^+(D)$ over all acyclic orientations D of G is clearly the degeneracy of G. Hence, Proposition 2.1 yields $\chi(G) \leqslant \delta^*(G) + 1$. Similarly, using acyclic orientation, one can prove $\mathrm{ch}(G) \leqslant \delta^*(G) + 1$.

In this section, we give some necessary conditions for a digraph D to have a proper colouring with $\Delta^+(D) + 1$ colours.

2.1 Kernels

Let D be a digraph. A set S of vertices is *antidominating* if every vertex v in $V(D) \setminus S$ dominates a vertex in S. An antidominating stable set is a *kernel*. In other words, a kernel of D is a stable set S such that $S \cup N^-(S) = V(D)$.

Let D be an acyclic digraph, and for every positive integer i, let S_i be the set of vertices coloured i in the greedy colouring with respect to D. Observe that S_1 is a kernel in D and more generally S_i is a kernel in $D - \bigcup_{j=1}^{i-1} S_j$. Hence greedily colouring with respect to D consists in iteratively colouring the vertices of a kernel of the digraph induced by the non-coloured vertices. This strategy of assigning a new colour to a kernel can be applied as long as the digraph induced by the non-coloured vertices has a kernel. This strategy certainly colours all vertices if the original digraph D is *kernel-perfect*, that is if every induced subdigraph of D has a kernel. It also applies to list colouring.

Lemma 2.2 (Bondy, Boppana and Siegel)**.** *Every kernel-perfect digraph D is $(d_D^+ + 1)$-chooosable. In particular, it is $(\Delta^+(D)+1)$-choosable.*

Proof. We prove the result by induction on the number of vertices. If $v(D) = 0$, there is nothing to prove. Now if $V(D)$ is nonempty, let L be a $(d_D^+ + 1)$-list assignment. We choose a colour c which appears in some list. Let X be the set of vertices x such that $c \in L(x)$. Since D is kernel-perfect, $D\langle X \rangle$ has a kernel S. Colour every vertex of S with colour c and delete colour c in the list of every vertex of $X \setminus S$, and apply the induction hypothesis on $D - S$. This is possible since every vertex of $X \setminus S$ has lost one colour and at least one inneighbour. Hence D has an L-colouring. □

It is easy to see that every acyclic digraph is kernel-perfect. (See Exercise 2.1.) Hence Lemma 2.2 implies Proposition 2.1. We cannot expect to improve the upper bound $\Delta(G)+1$ of this proposition by considering acyclic orientations because every acyclic orientation D of a regular graph G satisfies $\Delta^+(D) = \Delta(G)$. But it is tempting to use more balanced orientations.

To apply Lemma 2.2 non-acyclic orientations are only useful when they can be proven kernel-perfect. This is not always the case. For instance, if G is a complete graph of order at least 3, then no non-acyclic orientation is kernel-perfect (Exercise 2.2). However it is sometimes the case; in particular every orientation of a bipartite graph is kernel-perfect.

Theorem 2.3. *Bipartite digraphs are kernel-perfect.*

Proof. Because every subgraph of a bipartite digraph is bipartite, it suffices to prove that every bipartite digraph has a kernel. We prove this by induction on the number of vertices. Let D be a bipartite digraph. If D is strong, then each part of the bipartition is a kernel. If D is not strong, consider the kernel K of a terminal strong component of D. The union of K and a kernel of $D \setminus (K \cup N^-(K))$ is a kernel of D. □

Richardson [76] generalized this result by showing that digraphs without directed cycles of odd length are also kernel-perfect.

Together with Lemma 1.1 and Theorem 2.3, Lemma 2.2 yields the following result of Alon and Tarsi (See next subsection (2.2) for the original proof.).

Theorem 2.4 (Alon and Tarsi [6])**.** *Every bipartite graph G is $(\lceil \mathrm{Mad}(G)/2 \rceil + 1)$-choosable.*

Proof. By Lemma 1.1, G has an orientation D such that $\Delta^+ = \lceil \mathrm{Mad}(G)/2 \rceil$ and by Theorem 2.3, D is kernel-perfect. Thus by Lemma 2.2, it is $(\lceil \mathrm{Mad}(G)/2 \rceil + 1)$-choosable. □

Remark 2.5. Theorem 2.4 improves $\mathrm{ch}(G) \leqslant \delta^*(G)+1$ as $\mathrm{Mad}(G) \leqslant 2\delta^*(G)$.

Theorem 2.4 is sharp in the following sense.

Proposition 2.6. *For every positive integer k, there is a bipartite graph G with* $\mathrm{Mad}(G)/2 \leqslant k$ *which is not k-choosable.*

Proof. Consider the complete bipartite graph $K_{k^k,k}$ with bipartition (A,B) where $|A| = k^k$ and $|B| = k$.

If H is an induced subgraph of $K_{k^k,k}$ on $A' \cup B'$ with $A' \subset A$ and $B' \subset B$, then $e(H) \leqslant \sum_{a\in A'} d_H(a) \leqslant k|A'| \leqslant kv(H)$. Thus $\mathrm{Mad}(K_{k^k,k})/2 \leqslant k$.

Let us now show that $K_{k^k,k}$ is not k-choosable. Set $B = \{b_1,\dots,b_k\}$ and $A = \{a_{i_1,\dots,i_k} \mid 1 \leqslant i_j \leqslant k\}$. Let L be the list assignment defined by $L(b_i) = \{k(i-1)+j \mid 1 \leqslant j \leqslant k\}$ and $L(a_{i_1,\dots,i_k}) = \{i_j + (j-1)k \mid 1 \leqslant j \leqslant k\}$. Clearly G has no proper L-colouring. Indeed, assume that there exists such a colouring c. For every $1 \leqslant j \leqslant k$, let $i_j = c(b_j) - (j-1)k$. Then there is no colour in $L(a_{i_1,\dots,i_k})$ distinct from the colours of its neighbours, a contradiction. □

On the other hand, $\mathrm{Mad}(G)/2$ may be much larger than $\mathrm{ch}(G)$. For example consider the complete bipartite graph $K_{2^{k-1},2^{k-1}}$. Easily $\mathrm{Mad}(K_{2^{k-1},2^{k-1}})/2 = 2^{k-2}$ and $K_{2^{k-1},2^{k-1}}$ is k-choosable as shown by the following proposition.

Proposition 2.7. $\mathrm{ch}(K_{2^{k-1},2^{k-1}}) \leqslant k$.

Proof. Let (A,B) be the bipartition of $K_{2^{k-1},2^{k-1}}$. Let L be a k-list assignment of $K_{2^{k-1},2^{k-1}}$. Set $S = \bigcup_{v\in A\cup B} L(v)$. Let (S_A,S_B) be a random partition of S into two disjoint classes obtained by assigning colour of S independently either to S_A or S_B with probability $1/2$. A vertex v is called *bad* if either $v \in A$ and $L(v)\cap S_A = \emptyset$ or $v \in B$ and $L(v)\cap S_B = \emptyset$. The probability for a vertex to be bad is 2^{-k} so the expected number of bad vertices is 1. However, for the partitions $S_A = S$ and $S_B = \emptyset$ there are $2^{k-1} > 1$ bad vertices. So there is at least one partition (S_A,S_B) with no bad vertices. Choosing for each $a \in A$, $c(a) \in L(a)\cap S_A$ and for each $b \in B$, $c(b) \in L(b)\cap S_B$, we obtain a proper L-colouring of G. □

2.1.1 List edge-colouring bipartite graphs

Using the kernel approach, Galvin [40] proved that the List Colouring Conjecture restricted to bipartite graphs is true.

Theorem 2.8 (Galvin [40])**.** *Let G be a bipartite graph. Then* $\mathrm{ch}'(G) = \chi'(G) = \Delta(G)$.

The key step in Galvin's proof is to show that line graphs of bipartite graphs can be oriented in such a way that the maximum outdegree is not too high, and every induced subgraph has a kernel.

We need some preliminaries. Let $G = ((X,Y),E)$ be a bipartite graph. In the line graph $L(G)$, there is a clique K_v for each vertex v of G, the vertices of K_v corresponding to the edges of G incident to v. Each edge xy of G gives rise to a vertex of $L(G)$ which lies in exactly two of these cliques, namely K_x and K_y. We refer to K_v as an *X-clique* if $v \in X$, and a *Y-clique* if $v \in Y$.

There is a convenient way of visualizing this line graph $L(G)$. Because each edge of G is a pair xy, the vertex set of $L(G)$ is a subset of the cartesian product $X \times Y$. Therefore, in a drawing

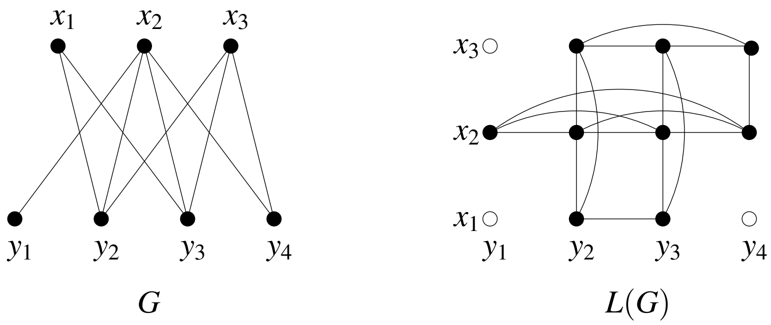


Figure 1: A bipartite graph G and a representation of its line graph $L(G)$.

of $L(G)$, we can place its vertices at appropriate lattice points of the $m \times n$ grid, where $m = |X|$ and $n = |Y|$, the rows of the grid being indexed by X and the columns by Y. Any two vertices which lie in the same row or column of the grid are adjacent in $L(G)$, and so the sets of vertices in the same row or column are cliques of $L(G)$, namely its X-cliques and Y-cliques, respectively. See Figure 1.

Lemma 2.9. *Let $G = ((X,Y),E)$ be a bipartite graph, and let D be an orientation of its line graph $L(G)$ in which each X-clique and each Y-clique induces a transitive tournament. Then D has a kernel.*

Proof. By induction on $e(G)$, the case $e(G) = 1$ being trivial. For $v \in V(G)$, denote by T_v the transitive tournament in D corresponding to v, and for $x \in X$, denote by t_x the sink of T_x. Set $K := \{t_x : x \in X\}$. Every vertex of $D - K$ lies in some T_x, and so dominates some vertex of K. Thus if the vertices of K lie in distinct Y-cliques, then K is a kernel of D.

Suppose, then, that the Y-clique T_y contains two vertices of K. One of these, say t_x, is not the source s_y of T_y, so $s_y \to t_x$. Set $D' := D - s_y$. Then D' is an orientation of the line graph $L(G \setminus e)$, where e is the edge of G corresponding to the vertex s_y of $L(G)$. Moreover, each clique of D' induces a transitive tournament. By induction, D' has a kernel K'. We show that K' is also a kernel of D. For this, it suffices to verify that s_y dominates some vertex of K'.

If $t_x \in K'$, then $s_y \to t_x$. On the other hand, if $t_x \notin K'$, then $t_x \to v$, for some $v \in K'$. Because t_x is the sink of its X-clique, v must lie in the Y-clique $T_y \setminus \{s_y\}$. But then s_y, being the source of T_y, dominates v. Thus K' is indeed a kernel of D. □

Proof of Theorem 2.8. Let $G = ((X,Y),E)$ be a bipartite graph with maximum degree k, and let $c : E(G) \to \{1,2,\dots,k\}$ be a k-edge-colouring of G. The colouring c induces a k-colouring of $L(G)$. We orient each edge of $L(G)$ joining two vertices of an X-clique from lower to higher colour, and each edge of $L(G)$ joining two vertices of a Y-clique from higher to lower colour as in Figure 2 (where the colour $c(x_i y_j)$ of the edge $x_i y_j$ is indicated inside the corresponfing vertex of $L(G)$). By Lemma 2.9, this orientation D is kernel-perfect. Moreover, $\Delta^+(D) = k - 1$. Thus, by Lemma 2.2, $L(G)$ is k-choosable, so G is k-edge-choosable. □

Slivnik [86] gave a direct alternative proof of Theorem 2.8. It also relies on the same orientation, but orientations and kernels are hidden.

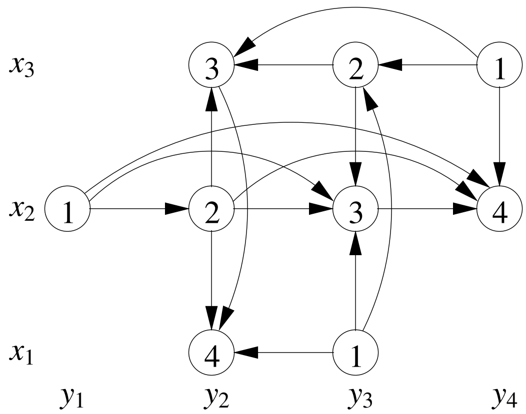


Figure 2: Orienting the line graph of a bipartite graph.

2.2 The graph polynomial, orientations and the Combinatorial Nullstellensatz

In this subsection, we define a natural polynomial associated to a graph, indeed so natural that is called the *graph polynomial*, and show how some of its coefficients may be expressed in terms of orientations. Next, we present an algebraic tool related to polynomials and apply it to obtain results on list colouring.

2.3 The graph polynomial

Let G be a graph with vertex set $\{v_1,\ldots,v_n\}$. The *graph polynomial* f_G of G is defined by $f_G(x_1,\ldots,x_n) := \prod\{(x_i - x_j) \mid i < j, v_iv_j \in E(G)\}$. This polynomial has been studied by several researchers, starting already with Petersen [72]. See also [84] and [63] for example. Note that a colouring c of G is proper if and only if $f_G(c(v_1),\ldots,c(v_n)) \neq 0$.

The coefficients of the monomials that appear in the standard representation of f_G as a linear combination of monomials can be expressed in terms of the orientations of G. While expanding the product $\prod\{x_1 - x_j) \mid i < j, v_iv_j \in E(G)\}$, to obtain a monomial one has too choose for each edge v_iv_j of E with $i < j$ either x_i or $-x_j$. This corresponds to choosing an orientation D of every edge v_iv_j $(i < j)$: We orient it from v_i to v_j if we choose x_i and from v_j to v_i if we choose $-x_j$. Hence each monomial we obtain is of the form $\prod_{i=1}^n x_i^{d_D^+(v_i)}$ with D an orientation of G. Furthermore, a monomial is preceded by the sign $+$ if the orientation D is *even* that is has an even number of arcs v_iv_j such that $i > j$. It is preceded by the sign $-$ if the orientation D is *odd* that is not even. For non-negative $d_1,\ldots d_n$ let $EO(d_1,\ldots d_n)$ and $OO(d_1,\ldots d_n)$ denote, respectively, the sets of all even and odd orientations of G in which $d^+(v_i) = d_i$, for $1 \leqslant i \leqslant n$. Then

$$f_G(x_1,\ldots,x_n) = \sum_{d_1,\ldots,d_n \geqslant 0} (|EO(d_1,\ldots,d_n)| - |OO(d_1,\ldots,d_n)|) \prod_{i=1}^n x_i^{d_i}. \tag{2}$$

Let $O(d_1,\ldots,d_n)$ be the set of orientations D of G such that $d_D^+(v_i) = d_i$, for $i \in \{1,\ldots,n\}$.

Let D be a fixed orientation in $O(d_1,\dots,d_n)$. For any orientation $D' \in O(d_1,\dots,d_n)$, let $\mathrm{diff}(D')$ be the spanning subdigraph of D with arc set $A(D) \setminus A(D')$. In other words, the arcs of $\mathrm{diff}(D')$ are the ones of D that are oriented in the opposite way in D'. Since the outdegree of every vertex in D' equals its outdegree in D, we deduce that $\mathrm{diff}(D')$ is an eulerian subdigraph of D. In addition, $\mathrm{diff}(D')$ is even if and only if either both D and D' are even or both are odd. The mapping $D' \mapsto \mathrm{diff}(D')$ is clearly a bijection between $O(d_1,\dots,d_n)$ and the set of eulerian spanning subdigraphs of D. In case D is even, it maps even orientations to even (eulerian) subdigraphs, and odd orientations to odd subdigraphs. Otherwise, it maps even orientations to odd subdigraphs, and odd orientations to even subdigraphs. In any case,

$$||EO(d_1,\dots,d_n)| - |OO(d_1,\dots,d_n)|| = |ee(D) - oe(D)|. \tag{3}$$

2.3.1 The Combinatorial Nullstellensatz

Let $\mathbb{F}$ be an arbitrary field, and let $\mathbb{F}[x]$ be the ring of polynomials in one variable. The well-known Factor Theorem states that a non-null polynomial of degree at most d has at most p roots. This theorem has been generalized to multivariate polynomials by Alon [7] in the so-called Combinatorial Nullstellensatz. This theorem has many useful applications in combinatorics, graph theory, and additive number theory (see [7]). The short proof we give is due to Michalek [67].

The *degree* of a multivariate polynomial P, denoted $\deg(P)$, is the degree of the polynomial $P(x,x,\dots,x)$ in x.

Theorem 2.10 (Combinatorial Nullstellensatz)**.** *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1,\dots,x_n)$ be a polynomial in $\mathbb{F}[x_1,\dots,x_n]$. Suppose that the degree $deg(P)$ of P is $\sum_{i=1}^n t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in P is non-zero. Then, if $S_1,\dots,S_n$ are subsets of $\mathbb{F}$ with $|S_i| > t_i$, there are $s_1 \in S_1,\dots,s_n \in S_n$ so that*

$$P(s_1,\dots,s_n) \neq 0.$$

Proof. By induction on the degree of P. If $\deg(P) = 0$ the theorem is trivial, as P is a non-zero constant. Suppose $\deg(P) > 0$ and P satisfies the assumptions of the theorem but $P(\vec{x}) = 0$ for every $\vec{x} \in S_1 \times \cdots \times S_n$. Without loss of generality $k_1 > 0$. We can write $s \in S_1$ and write

$$P = (x_1 - s)Q + R \tag{4}$$

where $Q \in \mathbb{F}[x_1,\dots,x_n]$ and $R \in \mathbb{F}[x_2,\dots,x_n]$. Since P vanishes on $\{s\} \times S_2 \times \cdots \times S_n$, R vanishes on $S_2 \times \cdots \times S_n$. Since P and R both vanish on $(S_1 \setminus \{s\}) \times S_2 \times \cdots \times S_n$, so does Q.

Note that the coefficient of $x_1^{k_1-1}x_2^{k_2}\cdots x_n^{k_n}$ in Q is non-zero and that $\deg(Q) = \sum_{i=1}^n k_i - 1 = \deg(P) - 1$. Hence we may apply the induction hypothesis to Q. So Q does not vanish on $(S_1 \setminus \{s\}) \times S_2 \times \cdots \times S_n$, a contradiction. □

2.3.2 The Alon-Tarsi Method

A digraph is *even* if it has an even number of arcs, otherwise it is *odd*. In particular, an empty digraph is an even digraph. We denote by $ee(D)$ and $oe(D)$ respectively the set of even and odd eulerian spanning subdigraphs of a digraph D.

Theorem 2.11 (Alon and Tarsi [6]). *Let D be a digraph. If $ee(D) \neq oe(D)$, then D is $(d_D^+ + 1)$-choosable. In particular, D is $(\Delta^+(D)+1)$-choosable.*

Proof. Let G be the underlying graph of D. For $1 \leqslant i \leqslant n$, set $d_i = d_D^+(v_i)$.

Hence if $ee(D) \neq oe(D)$ the coefficient of $\Pi_{i=1}^n x_i^{d_i}$ in f_G is non-zero. So by the Combinatorial Nullstellensatz for any list-assignment L such that $|L(v_i)| \geqslant d_i + 1$, there are $s_1 \dots, s_n$ such that $f_G(s_1, \dots, s_n) \neq 0$. Hence the colouring defined by $c(V_i) = s_i$ for $i \in \{1, \dots, n\}$ is a proper L-colouring. □

Observe that an acyclic digraph has no eulerian spanning subdigraph except the empty one. Therefore Theorem 2.11 also implies that $\chi(D) \leqslant \Delta^+(D) + 1$ for every acyclic digraph and so $\chi(G) \leqslant \delta^*(G) + 1$ for any graph G.

Theorem 2.4 also easily derives from Theorem 2.11.

Alternative proof of Theorem 2.4. Set $\Delta^+ = \lceil \mathrm{Mad}(G)/2 \rceil$. By Lemma 1.1, there is an orientation D of G in which the maximum outdegree is at most Δ^+. Since D contains no odd directed cycles (and in fact no odd cycles at all), $oe(D) = 0$ and $ee(D) \geqslant 1$ because the empty spanning subgraph is an even eulerian one. Hence $ee(D) \neq oe(D)$ and the result follows from Theorem 2.11. □

Häggkvist and Janssen [45] used the Alon-Tarsi method to show that the List Colouring Conjecture holds for complete graphs with an odd number of vertices, and that $\mathrm{ch}'(G) \leqslant \Delta(G) + O(\Delta(G)^{2/3}\sqrt{\log \Delta(G)})$.

Another interesting application of Theorem 2.11 has been obtained by Fleischner and Stiebitz [34], solving a problem raised by Du, Hsu and Hwang [25], as well as a strengthening of it suggested by Erdős.

Theorem 2.12 (Fleischner and Stiebitz [34]). *Let G be a graph on $3n$ vertices, whose edge set is the disjoint union of a hamiltonian cycle and n pairwise vertex-disjoint triangles. Then the choice number and the chromatic number of G are both 3.*

The proof is based on a subtle parity argument that shows that, if D is the digraph obtained from G by directing the hamiltonian cycle as well as each of the triangles cyclically, then $ee(D) - eo(D) \equiv 2 (mod 4)$. The result thus follows from Theorem 2.11.

Sachs [80] gave an elementary proof of the fact that the chromatic number of a 'cycle-plus-triangles' graph is 3-colourable. Fleischner and Sabidussi [35] considered the problem of 3-colourability of those 4-regular hamiltonian graphs whose edge set is the disjoint union of a hamiltonian cycle and pairwise vertex-disjoint non-selfcrossing cycles of constant length $\geqslant 4$. They showed that this problem is NP-complete.

In this section, we applied the kernel method and the Alon-Tarsi method to prove Theorem 2.4. However, each technique allows to establish results that cannot be proved by the other. For example, Theorem 2.8 proved by the kernel method cannot be proved by the Alon-Tarsi method. See Exercise 2.5.

On the other hand, Theorem 2.12 cannot be proved via kernels, since the 'cycle-plus-triangles' graphs are not kernel-perfect. Furthermore, contrary to the kernel method, the Alon-Tarsi method has been applied to different polynomials in order to establish results on various kinds of colourings. See for example Exercice 2.8.

2.4 Exercises

Exercise 2.1. Show that every acyclic digraph has a kernel. Deduce that every acyclic digraph is kernel-perfect and bi-kernel perfect.

Exercise 2.2. Show that a tournament is kernel-perfect if and only if it is acyclic.

Exercise 2.3. Show that every even cycle is 2-choosable in two ways:

1) using Theorem 2.2;

2) using Theorem 2.11.

Exercise 2.4.

A (directed) graph G is $(a:b)$-*choosable* if for any a-list assignment L of G, there exist sets $C(v)$, $v \in V(G)$, of size b such that $C(u) \cap C(v) = \emptyset$ for every edge $uv \in E(G)$.

1) Let D be a digraph, k a positive integer and L a $(kd^+ + k)$-list assignment of D. Show that if D contains no odd directed cycle, then there exist subsets $C(v) \subset L(v)$, where $|C(v)| = k$ for all $v \in V(D)$, and $C(u) \cap C(v) = \emptyset$ for every arc $uv \in A(D)$.

2) Deduce that an even cycle is $(2k:k)$-choosable, for every $k \geqslant 1$.

3) Show that if a connected graph G is not a complete graph nor an odd cycle, then $ch_{:k}(G) \leqslant k\Delta(G)$ for every $k \geqslant 1$.

4) Show that every d-degenerate graph is $(kd+k:k)$-choosable for every $k \geqslant 1$.

Exercise 2.5. Show that if D is an orientation of $L(K_{3,3})$ such that $\Delta^+(D) \leqslant 2$, then $ee(D) = eo(D)$.

Exercise 2.6. Let G be an interval graph.

1) Show that G has an acyclic orientation D with $\Delta^+(D) = \omega(G) - 1$.

2) Deduce that $\mathrm{ch}(G) = \chi(G) = \omega(G)$. (D. R. Woodall [95])

Exercise 2.7. The aim of this exercise is to show Brooks' Theorem for list colouring (Theorem 1.6) using the Alon-Tarsi method (Theorem 2.11). Let G be a connected graph with maximum degree Δ.

1) Assume that a vertex x has degree less than Δ.

a) Show that G has an acyclic orientation D such that $d_D^+(v) \leqslant \Delta - 1$ for every vertex v.

b) Deduce from Theorem 2.11 that $\chi(G) \leqslant \Delta(G)$.

2) Assume now that G is Δ-regular.

a) Show that G contains an even cycle C with at most one chord.

b) Show that G has an orientation D such that $D - C$ is acyclic, $C \to D - C$ and C is oriented in a cyclic way.

c) Deduce from Theorem 2.11 that $\chi(G) \leqslant \Delta(G)$. (Hladký, Král' and Schauz [57])

Exercise 2.8. Let p and q be two positive integers with $p \geqslant q$. A (p,q)-*colouring* of a graph G is a colouring f of the vertices of G with colours $\{0,1,\dots,p-1\}$ such that $q \leqslant |f(u)-f(v)| \leqslant p-q$ for all $uv \in E(G)$. A list assignment L is a t-(p,q)-*list-assignment* if $L(v) \subseteq \{0,\dots,p-1\}$ and $|L(v)| \geqslant tq$ for each vertex $v \in V$. The graph G is (p,q)-L-*colourable* if there exists a (p,q)-L-*colouring* c, i.e. c is both a (p,q)-colouring and an L-colouring. For any real number $t \geqslant 1$, the graph G is t-(p,q)-*choosable* if it is (p,q)-L-colourable for every t-(p,q)-list-assignment L. Last, G is *circularly* t-*choosable* if it is t-(p,q)-choosable for any p,q. The *circular list chromatic number* or *circular choice number* of G is

$$\mathrm{cch}(G) := \inf\{t \geqslant 1 : G \text{ is circularly } t\text{-choosable}\}.$$

The aim of this exercise is to prove $\mathrm{cch}(C_{2n}) = 2$ for any $n \geqslant 2$, a result due to Norine [71].

1) Show that $\mathrm{cch}(G) \geqslant 2$ for every non-empty graph G.

2) Considering the polynomial of $\mathbb{C}[x_1,\dots,x_n]$

$$P(x_1,x_2,\dots,x_{2n}) = \prod_{j=1}^{2n} \prod_{k=-q+1}^{q-1} (x_j - \exp(2\pi ik/p)x_{j+1})$$

where i denotes the square root of -1, prove that C_{2n} is circularly 2-choosable.

Exercise 2.9. Let $G = (V,E)$ be a graph and let f be a function assigning to eahc $v \in V$ a set of $f(v)$ integers in $\{0,\dots,d(v)\}$. An f-*factor* of G is a subgraph of G in which $d(v) \in f(v)$ for all $v \in V$.

Considering the polynomial over $\mathbb{R}$ in the variables x_e, $e \in E$,

$$\prod_{v \in V} \prod_{c \in f(v)^c} \left(\sum_{e \ni v} x_e - c \right)$$

where $f(v)^c = \{0,\dots,d(v)\} \setminus f(v)$, show that if $|f(v)| > \lceil d(v)/2 \rceil$ for every $v \in V$, then G has an f-factor. (Shirazi and Verstraëte [85])

Exercise 2.10. Let p be a prime and $G=(V,E)$ a graph with average degree bigger than $2p-2$ and maximum degree at most $2p-1$.

For every vertex v and edge e, let $a_{v,e}=1$ if v is incident to e and $a_{v,e}=0$ otherwise. Considering the polynomial over $GF(p)$ in the variables x_e, $e\in E$,

$$\prod_{v\in V}\left(1-\left(\sum_{e\in E}a_{v,e}x_e\right)^{p-1}\right)-\prod_{e\in E}(1-x_e)$$

show that G contains a p-regular subgraph. (N. Alon, S. Friedland and G. Kalai [4])

3 Properties of k-chromatic digraphs

Recall that a (proper) *colouring* of a digraph D is simply a colouring of its underlying graph G, and its *chromatic number* $\chi(D)$ is defined to be the chromatic number $\chi(G)$ of G. Why, then, consider colourings of digraphs? It turns out that the chromatic number of a digraph provides interesting information about its subdigraphs.

3.1 Gallai–Roy Theorem and related results

One of the most well known connections between orientations and colourings is the so called Gallai–Roy Theorem. But in fact in the 1960s it was published in four different languages by four different authors: Gallai [38] in English, Hasse [50] in German, Roy [79] in French and Vitaver [92] in Russian.

Theorem 3.1 (Gallai [38] –Hasse [50] – Roy [79] – Vitaver [92]). GALLAI–ROY THEOREM
The order of the longest path of a digraph D is at least $\chi(D)$.

There are many proofs of this theorem. The proof we give here is due to El-Sahili and Kouider [28]. It is based on some order on spanning out-forests, which will be useful later. Another proof, using the concept of median order which will be studied later, is outlined in Exercise 3.10.

Let F be a spanning out-forest of D. The *level* of x is the number of vertices of a longest directed path of F ending at x. For instance, the level 1 vertices are the roots of the out-arborescences of F. We denote by F_i the set of vertices with level i in F. A vertex y is a *descendant* of x in F if there is a directed path from x to y in F.

If there is an arc xy in D from F_i to F_j, with $i\geqslant j$, and x is not a descendant of y, then the out-forest F' obtained by adding xy and removing the arc of F with head y (if such exists that is if $j>1$) is called an *elementary improvement* of F. An out-forest F' is an *improvement* of F if it can be obtained from an out-forest F by a sequence of elementary improvements. The key-observation is that if F' is an improvement of F then the level of every vertex in F' is at least its level in F. Moreover, at least one vertex of F has its level in F' strictly greater than its level in F. Thus, one cannot perform infinitely many improvements. A spanning out-forest F is *final* if there is no elementary improvement of F.

The following proposition follows immediately from the definition of final spanning out-forest:

Proposition 3.2 (El-Sahili and Kouider [28]). *Let D be a digraph and F a final spanning out-forest of D. If a vertex $x \in F_i$ dominates in D a vertex $y \in F_j$ for $j \leqslant i$, then x is a descendant of y in F. In particular, every level of F is a stable set in D.*

Proof of Theorem 3.1. Consider a final spanning out-forest of a k-chromatic digraph D. Since every level is a stable set by Proposition 3.2, there are at least k levels. Hence D contains a directed path of order at least k. □

The chromatic number of a graph may then be defined in terms of the lengths of a longest path in its orientations. Let $\lambda(D)$ denote the length of a longest directed path in the digraph D.

Corollary 3.3. *For any graph G,*

$$\chi(G) = \min\{\lambda(D) \mid D \textit{ orientation of } G\}.$$

Proof. As $\chi(G) = \chi(D)$, by Theorem 3.1, $\chi(G) = \min\{\lambda(D) \mid D \text{ orientation of } G\}$.

Let $(S_1, \dots, S_{\chi(G)})$ be a partition of $V(G)$ into stable sets that corresponds to a proper $\chi(G)$-colouring of G. For any $(i, j) \in \{1, \dots, \chi(G)\}^2$ with $i < j$, orient all the edges between S_i and S_j form S_i to S_j. Doing so we obtain an orientation of G in which every directed path has at most $\chi(G)$ vertices. □

Remark 3.4. The above proof shows that $\min\{\lambda(D) \mid D \text{ orientation of } G\}$ is always attained by an acyclic orientation. But it may be attained by other orientations.

Not only is there an upper bound for the chromatic number of a graph G in terms of the length of a longest path in an orientation D of G, an upper bound can be given with the aid of orientations and the cycles of G. Let D be an acyclic orientation of a graph G that is not a forest. For each cycle C of G, there are then $a(C)$ edges of C oriented in one direction and $b(C)$ edges oriented in the opposite direction for some positive integers $a(C)$ and $b(C)$ with $a(C) \geqslant b(C)$. Let

$$r(D) := \max\left\{\frac{a(C)}{b(C)} \,\middle|\, C \text{ is a cycle of } G\right\}.$$

The following theorem due to Minty [68] relates the chromatic number $\chi(G)$ to $r(D)$ for some acyclic orientation D of G.

Theorem 3.5 (Minty [68]). *Let G be a graph that is not a forest. Then $\chi(G) \leqslant k$ for some integer $k \geqslant 2$ if and only if there exists some acyclic orientation D of G such that $r(D) \leqslant k - 1$.*

The existence of an acyclic orientation D such that $r(D) \leqslant k - 1$ is easy (Exercise 3.3). The converse is more difficult.

Corollary 3.6. *Let G be a graph that is not a forest. Then*

$$\chi(G) = \min\{1 + \lceil r(D) \rceil \mid D \text{ orientation of } G\}.$$

3.1.1 Rédei's Theorem

Because a tournament of order n has chromatic number n, the Gallai–Roy Theorem is a generalization of the well-known Rédei's Theorem.

Theorem 3.7 (Rédei [74]). RÉDEI'S THEOREM
Every tournament contains a hamiltonian directed path.

This theorem has many easy and elementary proofs. See Exercise 3.1. An interesting generalization is a theorem of Rédei [74] on the parity of the number of hamiltonian directed paths; it was from this result that it was originally deduced.

Theorem 3.8 (Rédei [74]). *Every tournament contains an odd number of hamiltonian directed paths.*

The proof of Theorem 3.8 is established by means of a proof technique known as *Inclusion-Exclusion Principle* or *Möbius Inversion Formula*, an inversion formula with applications throughout mathematics. We present here a simple version which suffices for our purpose. We refer the interested reader to Chapter 21 of Handbook of Combinatorics [41].

Lemma 3.9. MÖBIUS INVERSION FORMULA
Let T be a finite set and $f : 2^T \to \mathbb{R}$ a real-valued function defined on the subsets of T. Define the function $g : 2^T \to \mathbb{R}$ by $g(X) = \sum_{Y|X\subseteq Y\subseteq T} f(Y)$. Then

$$f(X) = \sum_{Y|X\subseteq Y\subseteq T} (-1)^{|Y|-|X|} g(Y).$$

Proof. By the Binomial Theorem,

$$\sum_{XY|\subseteq Y\subseteq Z} (-1)^{|Y|-|X|} = \sum_{k=|X|}^{|Z|} \binom{|Z|-|X|}{k-|X|} (-1)^{k-|X|} = (1-1)^{|Z|-|X|}$$

which is equal to 0 if $X \subseteq Z$, and to 1 if $X = Z$. Therefore,

$$\begin{aligned} f(X) &= \sum_{Z|X\subseteq Z\subseteq T} f(Z) \sum_{Y|X\subseteq Y\subseteq Z} (-1)^{|Y|-|X|} \\ &= \sum_{Y|X\subseteq Y\subseteq T} (-1)^{|Y|-|X|} \sum_{Z|Y\subseteq Z\subseteq T} f(Z) = \sum_{Y|X\subseteq Y\subseteq T} (-1)^{|Y|-|X|} g(X) \end{aligned}$$

□

Proof of Theorem 3.8. Let T be a tournament with vertex set $V(T) = \{1,2,\dots,n\}$. For any permutation σ of $V(T)$, let $A_\sigma = A(T) \cap \{\sigma(i)\sigma(i+1) \mid 1 \leqslant i \leqslant n-1\}$. Then A_σ induces a subdigraph of T each of whose components is a directed path.

For any subset X of $A(T)$, let us define $f(X) = |\{\sigma \in \mathcal{S}_n \mid X = A_\sigma\}|$ and $g(X) = |\{\sigma \in \mathcal{S}_n \mid X \subset A_\sigma\}|$. Then $g(X) = \sum_{Y | X \subseteq Y \subseteq A(T)} f(Y)$, so by the Möbius Inversion Formula

$$f(X) = \sum_{Y | X \subseteq Y \subseteq A(T)} (-1)^{|Y|-|X|} g(Y).$$

Observe that $g(Y) = r!$ if and only if the spanning subdigraph of D with arc-set Y is the disjoint union of r directed paths. Thus $g(Y)$ is odd if and only if Y induces a hamiltonian directed path of D. Hence, defining $h(X) = |\{H \in \mathcal{H} \mid X \subset A(H)\}|$ with $\mathcal{H}$ the set of hamiltonian directed paths of D, we obtain

$$f(X) \equiv \sum_{\{H \in \mathcal{H} \mid X \subset A(H)\}} (-1)^{n-1-|X|} \equiv h(X) \pmod 2.$$

The theorem is true for transitive tournaments as there is a unique hamiltonian directed path. Since any tournament of order n may be obtained from the transitive tournament on n vertices by reversing the orientation of appropriate arcs, it suffices to prove that the parity of the number of hamiltonian directed paths $h(T)$ is unaltered by the reversal of any one arc a.

Let T' be the tournament obtained from T by reversing a. Then $h(T') = h(T) + f(\{a\}) - h(\{a\})$. Since $f(\{a\}) \equiv h(\{a\}) \pmod 2$, we have $h(T') \equiv h(T) \pmod 2$. □

3.1.2 Relation to Gallai-Milgram Theorem

Gallai–Roy Theorem (3.1) bears a striking formal resemblance with another celebrated result, namely Gallai–Milgram Theorem. By interchanging the roles of stable sets and directed paths, one theorem is transformed into the other: directed paths become stable sets, and vertex colourings (which are partitions into stable sets) become path partitions, where by *path partition*, we mean a covering of the vertex set of a digraph by disjoint directed paths. A path partition with the fewest paths is called an *optimal path partition*. The number of directed paths in an optimal path partition of a digraph D is denoted by $\pi(D)$. Recall that $\alpha(D)$ is the stability number of D.

Theorem 3.10 (Gallai and Milgram [39]). GALLAI–MILGRAM THEOREM
Every digraph D has a path partition in at most $\alpha(D)$ paths, that is $\pi(D) \leqslant \alpha(D)$.

[39] actually proved a somewhat stronger theorem. A directed path P and a stable set S are said to be *orthogonal* if they have exactly one common vertex. By extension, a path partition $\mathcal{P}$ and stable set S are *orthogonal* if each path in $\mathcal{P}$ is orthogonal to S.

Theorem 3.11 (Gallai and Milgram [39]). *Let $\mathcal{P}$ be an optimal path partition of a digraph D. Then there is a stable set S in D which is orthogonal to $\mathcal{P}$.*

Note that the Gallai–Milgram Theorem is an immediate consequence of Theorem 3.11 because $\pi = |\mathcal{P}| \leqslant |S| \leqslant \alpha$. Theorem 3.11 is established by means of a nice inductive argument. See Exercise 3.4.

A possible common generalization of the Gallai–Roy and Gallai–Milgram theorems was proposed by Linial [65]. A partial k-colouring $\mathcal{C}$ of a digraph D is *optimal* if the number of coloured vertices, $\sum_{C\in\mathcal{C}}|C|$, is as large as possible. Let $\mathcal{P}$ be a k-optimal path partition and $\mathcal{C}$ an optimal k-colouring of a digraph D. Define

$$\pi_k(D) := \sum_{P\in\mathcal{P}} \min\{v(P),k\} \quad \text{and} \quad \alpha_k(D) := \sum_{C\in\mathcal{C}} |C|\,.$$

In particular, $\pi_1 = \pi$, the number of directed paths in an optimal path partition of D, and $\alpha_1 = \alpha$. Linial's Conjecture asserts that $\pi_k \leqslant \alpha_k$ for all digraphs D and all positive integers k.

Conjecture 3.12 (Linial [65])**.** For every positive integer k and every digraph D, $\pi_k(D) \leqslant \alpha_k(D)$.

A stronger conjecture than Linial's one was proposed by Berge [9]. See Exercise 3.5.

Let k be a positive integer. A path partition $\mathcal{P}$ is *k-optimal* if it minimizes the function $\sum\{\min\{v(P),k\} : P\in\mathcal{P}\}$, and a *partial k-colouring* of a graph or digraph is a family of k disjoint stable sets. In particular, a 1-optimal path partition is one that is optimal, and a partial 1-colouring is simply a stable set.

The concept of orthogonality of paths and stable sets is extended as follows. A path partition $\mathcal{P}$ and partial k-colouring $\mathcal{C}$ are *orthogonal* if every directed path $P\in\mathcal{P}$ meets $\min\{v(P),k\}$ different colour classes of $\mathcal{C}$. We can now state the conjecture proposed by Berge.

Conjecture 3.13 (Berge [9])**.** PATH PARTITION CONJECTURE
Let D be a digraph, k a positive integer, and $\mathcal{P}$ a k-optimal path partition of D. Then there is a partial k-colouring of D which is orthogonal to $\mathcal{P}$.

The Path Partition Conjecture has been proved for $k = 1$ by Linial [64] and for $k = 2$ by Berger and Hartman [10]. It has also been established for acyclic digraphs, by Aharoni et al. [3] and Cameron [19], and for digraphs containing no directed path with more than k vertices, by Berge [9]. We refer the reader to the survey by Hartman [49] for a full discussion of this conjecture and of related questions.

Gallai–Roy Theorem (3.1) would be implied by the following conjecture and an immediate induction.

Conjecture 3.14 (Laborde, Payan and Xuong [62], 1982)**.** Every digraph has a stable set meeting every longest directed path.

For digraphs with stability number 1, i. e. tournaments, the conjecture is true because they have a hamiltonian directed path. Thus the removal of any vertex decreases the length of a longest directed path. Havet [52] verified Conjecture 3.14 for digraph with stability number 2 using a result of Chen and Manalastas [22] asserting that every every strong digraph with stability number 2 has a hamiltonian directed path.

Analogously, Gallai-Milgram Theorem 3.10 would be implied by the fact that every digraph has a directed path meeting all maximum stable sets. However, using a clever probabilistic argument, Fox and Sudakov [37] showed that, as conjectured by Hahn and Jackson [47], it is false in the following strong sense. For each positive integer k, there is a digraph D with stability number k such that deleting the vertices of any $k-1$ directed paths in D leaves a digraph with stability number k.

3.2 Unavoidable and universal digraphs

A digraph is *k-universal* if it is contained in every k-chromatic digraph and simply *universal* if there exists some k such that it is k-universal. Similarly, a digraph is *k-unavoidable* if it is contained in every tournament of order k and simply *unavoidable* if there exists some k such that it is k-unavoidable. Trivially, if a digraph is k-universal, then it is k-unavoidable.

Gallai–Roy Theorem (resp. Redei's Theorem) states that the directed path of order k is k-universal (resp. k-unavoidable). A natural question is which digraphs are k-universal (resp. k-unavoidable)? Dually, for each digraph D one can wonder if it is universal (resp. unavoidable) and, if yes, what is the minimum integer $\mathrm{univ}(D)$ (resp. $\mathrm{unvd}(D)$) such that D is $\mathrm{univ}(D)$-universal (resp. $\mathrm{unvd}(D)$-unavoidable).

3.2.1 Unavoidable digraphs

Because the transitive tournaments are acyclic, every digraph containing a directed cycle is not unavoidable. On the other hand, every acyclic digraph of order k is 2^{k-1}-unavoidable.

Theorem 3.15. *Every acyclic digraph of order k is 2^{k-1}-unavoidable.*

Proof. As every acyclic digraph of order k is a subdigraph of the transitive tournament TT_k, it suffices to prove the result for TT_k.

We prove it by induction on k, the result holding trivially if $k=1$. Let $k>1$ and let T be a tournament of order 2^k. In T there is a vertex v of outdegree at least 2^{k-1}. By the induction hypothesis, $T\langle N^+(v)\rangle$ has a transitive subtournament T' of order $k-1$. Thus $T\langle V(T')\cup\{v\}\rangle$ is a transitive tournament of order k. □

The above result is almost tight for transitive tournaments as shown by the following result due to Erdős and Moser [33].

Theorem 3.16 (Erdős and Moser [33]). *There exists a tournament on $2^{(k-1)/2}$ vertices which contains no TT_k.*

Proof. The proof is probabilistic and uses the First Moment Method. Set $n=2^{(k-1)/2}$ and let T be a random tournament on n vertices.

For ordered k-tuple $(v_1,\dots,v_k)$ the probability that $T\langle\{v_1,\dots,v_k\}\rangle$ is a transitive tournament with hamiltonian directed path $(v_1,v_2,\dots,v_k)$ is $\left(\frac{1}{2}\right)^{\binom{k}{2}}$. Hence the expected number of transitive subtournaments of order k is

$$\frac{n!}{(n-k)!}\left(\frac{1}{2}\right)^{\binom{k}{2}} < n^k\left(\frac{1}{2}\right)^{\binom{k}{2}} \leqslant 1$$

because $n\leqslant 2^{(k-1)/2}$. Hence by the First Moment Principle, there exists a tournament of order n with less than 1 i.e. no transitive subtournament of order k. □

More generally, for every acyclic digraph D with k vertices and m arcs one shows that $\mathrm{unvd}(D) > 2^{\frac{m}{k}}$. See Exercice 3.8. This gives a meaningful lower bound for digraphs with sufficiently many arcs, namely at least $k \log k$ arcs.

For transitive tournaments, Erdős and Moser [33] ask for the value of $\mathrm{unvd}(TT_k)$.

Problem 3.17. Erdős and Moser [33] What is $\mathrm{unvd}(TT_k)$?

Clearly, $\mathrm{unvd}(TT_1) = 1$, $\mathrm{unvd}(TT_2) = 2$ and $\mathrm{unvd}(TT_3) = 4$. Also $\mathrm{unvd}(TT_4) = 8$ because the Paley tournament (depicted on Figure 3) contains no TT_4. Moreover Reid and Parker [75] showed that $\mathrm{unvd}(TT_5) = 14$ and $\mathrm{unvd}(TT_6) = 28$ and Sanchez-Flores [83] showed $\mathrm{unvd}(TT_7) \leqslant 54$. A similar induction as in the proof of Theorem 3.15 yields that $\mathrm{unvd}(TT_k) \leqslant 54 \times 2^{k-7}$ if $k \geqslant 7$. In addition, for $1 \leqslant k \leqslant 6$ it has been shown [75, 82] that there is a unique tournament of order $\mathrm{unvd}(TT_k) - 1$ that contains no TT_k.

3.2.2 Universal digraphs

Since there exist k-chromatic graphs with arbitrarily large girth (Theorem 1.2), universal digraphs must be oriented trees. Conversely, Burr [17] proved that every oriented tree is universal. In fact, he showed that every oriented tree of order k is $(k-1)^2$-universal. We will no show a slight improvement of this result due to Addario-Berry et al. [2].

Theorem 3.18 (Addario-Berry et al. [2])**.** *Every oriented tree on k vertices is $(k^2/2 - k/2 + 1)$-universal.*

The proof relies on the notion of kernel and its dual notion *antikernel*. A set S is *antidominating* if every vertex v in $V(D) \setminus S$ dominates a vertex in S. A dominating stable set is called an *antikernel*. In other words, an antikernel of D is a stable set S such that $S \cup N^+(S) = V(D)$. If every induced subdigraph of D has a kernel and an antikernel, then D is said to be *bikernel-perfect*. Several classes of bikernel-perfect digraphs are known: symmetric digraphs, acyclic digraphs are bikernel-perfect.

The following lemma shows that one can find large trees in highly chromatic bikernel-perfect digraphs.

Lemma 3.19 (Addario-Berry et al. [2])**.** *Every oriented tree of order k is contained in every k-chromatic bikernel-perfect digraph.*

Proof. Let us prove the result by induction on k, the result being trivially true if $k = 1$.

Let T be an oriented tree of order k and D be a k-chromatic bikernel-perfect digraph. Let v be a leaf of T and w its unique neighbour in T. By directional symmetry, we may assume that $v \to w$. Since D is bikernel-perfect, T has a kernel S. The digraph $D - S$ has chromatic number at least $(k-1)$, so by induction it contains a copy T' of $T - v$. Now by definition of kernel, the vertex w' in T' corresponding to w is dominated by a vertex v' of S. Hence D contains T. $\square$

Proof of Theorem 3.18. Let $f(k) = (k^2/2 - k/2 + 1)$. We have $f(k) = f(k-1) + k - 1$. Let us prove the result by induction on k. The result holding trivially when $k = 1$.

Suppose now that $k > 1$. Let D be an $f(k)$-chromatic digraph and T be an oriented tree of order k. Let A be a maximal acyclic induced subdigraph of D. If $\chi(A) \geqslant k$, then by Theorem 3.19, A contains T, so D contains T. If $\chi(A) \leqslant k-1$, then $\chi(D-A) \geqslant f(k)-(k-1) = f(k-1)$. Let v be a leaf of T. The digraph $D-A$ contains $T-v$. Now, by maximality of A, for every vertex x of $D-A$, there are vertices y and z of A such that xy and zx are arcs. So we can extend $T-v$ to T by adding a vertex in A. □

Since every oriented tree is universal, it is natural to ask for the value of $\mathrm{univ}(T)$ for every oriented tree T. Furthermore, what is

$$\mathrm{univ}(k) = \max\{\mathrm{univ}(T) \mid T \text{ oriented tree of order } k\} \ ?$$

Theorem 3.18 yields $\mathrm{univ}(k) \leqslant k^2/2 - k/2 + 1$. However this bound is believed to be far from tight. Burr [17] conjectured that $\mathrm{univ}(k) = 2k-2$.

Conjecture 3.20 (Burr [17])**.** Every oriented tree with $k > 1$ vertices is $(2k-2)$-universal.

This conjecture is tight. In a regular tournament R of order $2k-3$ all vertices have outdegree $k-2$. Hence R does not contain the outstar S_k^+, the oriented tree of order k consisting of a vertex dominating all the others. Hence S_k^+ is not $(2k-3)$-unavoidable, and so not $(2k-3)$-universal.

In view of Lemma 3.19, an approach to improve the upper bound on $\mathrm{univ}(k)$ would be to prove that every digraph with not too big chromatic number contains a bikernel-perfect k-chromatic digraph.

Problem 3.21 (Addario-Berry et al. [2])**.** What is the minimum integer $g(k)$ such that every $g(k)$-chromatic digraphs has an acyclic k-chromatic digraph?

Proposition 3.22. *Every k^2-chromatic digraph contains a k-chromatic acyclic subdigraph.*

Proof. Let D be a k^2-chromatic digraph. Let $v_1, v_2, \dots, v_n$ be any linear order of the vertices of D. Let D_1 and D_2 be the digraphs with vertex set $V(D)$ and arc sets $A(D_1) = \{v_iv_j \in A(D), i < j\}$ and $A(D_2) = \{v_iv_j \in A(D), i > j\}$. Clearly, D_1 and D_2 are acyclic and $\chi(D_1) \times \chi(D_2) \geqslant \chi(D)$. Hence either D_1 or D_2 has chromatic number at least k. □

3.3 Oriented trees in k-chromatic digraphs

3.3.1 Oriented trees in tournaments

Conjecture 3.23 (Sumner, 1972)**.** Every oriented tree with $k > 1$ vertices is $(2k-2)$-unavoidable.

The first linear bound was given by Häggkvist and Thomason [46]. Refining an idea of Havet and Thomassé [55], El Sahili [27] proved that every oriented tree of order k ($k \geqslant 2$) is $(3k-3)$-unavoidable. Recently, Kühn, Mycroft and Osthus [61] proved that Sumner's conjecture is true for all sufficiently large k. Their complicated proof makes use of the directed version of the Regularity Lemma.

We now present the method used by Havet and Thomassé [55]. It is based on the concept of median order. A *median order* of a digraph D is a linear order $v_1, v_2, \dots, v_n$ of its vertex set such that $|\{(v_i, v_j) : i < j\}|$ (the number of arcs directed from left to right) is as large as possible. In the case of a tournament, such an order can be viewed as a ranking of the players which minimizes the number of upsets (matches won by the lower-ranked player). As we shall see, median orders of tournaments reveal a number of interesting structural properties.

Let us first note two basic properties of median orders of tournaments (Exercise 3.9). Let T be a tournament and $v_1, v_2, \dots, v_n$ a median order of T. Then, for any two indices i, j with $1 \leqslant i < j \leqslant n$:

(M1) the interval $v_i, v_{i+1}, \dots, v_j$ is a median order of the induced subtournament $T\langle\{v_i, v_{i+1}, \dots, v_j\}\rangle$,

(M2) vertex v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \dots, v_j$, and vertex v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \dots, v_{j-1}$.

In particular, each vertex v_i, $1 \leqslant i < n$, dominates its successor v_{i+1}. The sequence $(v_1, v_2, \dots, v_n)$ is thus a hamiltonian directed path, providing an alternative proof of Rédei's Theorem (3.7).

Theorem 3.24 (Havet and Thomassé [55]). *Every tournament of order $2k-2$ contains every arborescence of order k.*

Proof. By directional duality, it suffices to prove for out-arborescences.

Let $v_1, v_2, \dots, v_{2k-2}$ be a median order of a tournament T on $2k-2$ vertices, and let A be an out-arborescence on k vertices. Consider the intervals $v_1, v_2, \dots, v_i$, $1 \leqslant i \leqslant 2k-2$. We show, by induction on k, that there is a copy of A in T whose vertex set includes at least half the vertices of any such interval.

This is clearly true for $k = 1$. Suppose, then, that $k \geqslant 2$. Delete a leaf y of A to obtain an out-arborescence A' on $k-1$ vertices, and set $T' := T - \{v_{2k-3}, v_{2k-2}\}$. By (M1), $v_1, v_2, \dots, v_{2k-4}$ is a median order of the tournament T', so there is a copy of A' in T' whose vertex set includes at least half the vertices of any interval $v_1, v_2, \dots, v_i$, $1 \leqslant i \leqslant 2k-4$. Let x be the predecessor of y in A. Suppose that x is located at vertex v_i of T'. In T, by (M2), v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \dots, v_{2k-2}$, thus at least $k-1-i/2$ of these vertices. On the other hand, A' includes at least $(i-1)/2$ of the vertices $v_1, v_2, \dots, v_{i-1}$, thus at most $k-1-(i+1)/2$ of the vertices $v_{i+1}, v_{i+2}, \dots, v_{2k-2}$. It follows that, in T, there is an outneighbour v_j of v_i, where $i+1 \leqslant j \leqslant 2k-2$, which is not in A'. Locating y at v_j, and adding the vertex y and arc (x, y) to A', we now have a copy of A in T. It is readily checked that this copy of A satisfies the required additional property. $\square$

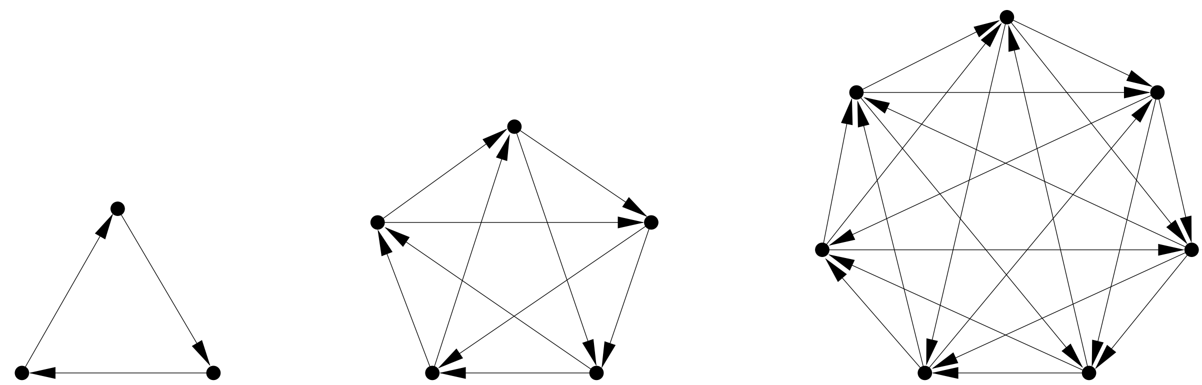
The same method can be easily adapted to prove that every oriented tree of order k is $(4k-4)$-unavoidable. (Exercise 3.11). El Sahili [27] used it in a clever way to show that every oriented tree of order k is $(3k-3)$-unavoidable. (Exercise 3.12).

Havet and Thomassé (see [54]) made a conjecture implying Sumner's one.

Conjecture 3.25 (Havet and Thomassé, 1996). Every oriented tree with k vertices and ℓ leaves is $(k+\ell-1)$-unavoidable.

As an evidence to Conjecture 3.25 Häggkvist and Thomason [46] proved that oriented tree with k vertices and ℓ leaves is $(k+2^{512\ell^3})$. Havet [51] proved that a large family of oriented trees verify the conjecture. For $\ell = 2$, Conjecture 3.25 follows from a result of Thomason [89] stating that every oriented path of order k is $(k+1)$-unavoidable (See [56] for a short inductive proof.). Ceroi and Havet [21] proved the case $\ell = 3$.

Thomason's result cannot be improved since the antidirected paths of order 3, 5, and 7 are not contained in the 3-cycle C_3, the regular tournament on 5 vertices R_5, and the Paley tournament on 7 vertices P_7, respectively. See Figure 3. These pairs (P,T) of antidirected path and tounament of the same order, such that T does not contain P are known as Grünbaum's exceptions. Havet and Thomassé [56] proved that they are the only ones, thus answering a



C_3 R_5 P_7

Figure 3: The tournaments of Grünbaum's exceptions

conjecture of Roesenfeld [77].

Theorem 3.26 (Havet and Thomassé [56])**.** *Let T and P be a tournament and an oriented path of the same order. Then T contains P unless (P,T) is one of Grünbaum's exceptions.*

A similar phenomenon appears when $\ell = 3$. Ceroi and Havet [21] actually proved that every oriented tree of of order k with three leaves is $(k+1)$-unavoidable except the outstar S_3^+ and its converse S_3^-.

3.3.2 Universality of oriented paths

It might be possible that Conjecture 3.25 extends to universality.

Conjecture 3.27. Every oriented tree with k vertices and ℓ leaves is $(k+\ell-1)$-universal.

This conjecture would imply Burr's Conjecture. However, very little is known about this conjecture. Regarding oriented paths, it suggests that every oriented path of order k is $(k+1)$-universal. In fact, Bondy conjectured that except a finite number of exceptions, they should be k-universal.

Conjecture 3.28 (Bondy)**.** For sufficiently large k, every oriented path of order k is k-universal.

El-Sahili proved [26] that every path of order 4 is 4-universal and that the antidirected path of order 5 is 5-universal. The only generic results so far regards paths with two blocks. The *blocks* of a path are its maximal directed subpaths.

Theorem 3.29 (Addario-Berry, Havet and Thomassé [1])**.** *If $k \geqslant 4$, then every oriented k-path with two blocks is k-universal.*

Remark 3.30. The condition $k \geqslant 4$ in this theorem is necessary, as the 3-cycle C_3 does not contain the antidirected path of order 3.

The proof of Theorem 3.29 uses the concept of final spanning out-forest and the notion of *good directed cycle*. We now show a slightly weaker statement than Theorem 3.29 which emphasizes the use of final spanning forests. Good directed cycles are defined and used to prove the restriction of Theorem 3.29 to strong digraphs in Subsection 3.4.2.

Theorem 3.31 (El-Sahili and Kouider [28])**.** *Every oriented k-path with two blocks is $(k+1)$-universal.*

By directional symmetry we may consider that an oriented path with two blocks is an oriented path of order k_1+k_2+1 starting with k_1 forward arcs and followed by k_2 backward arcs for some $k_1 \geqslant 1$ and $k_2 \geqslant 1$. Let us denote such a path by $P(k_1,k_2)$. We shall prove that $P(k_1,k_2)$ is (k_1+k_2+1)-universal.

To do so, we need the following lemma.

Lemma 3.32 (El-Sahili and Kouider [28])**.** *Let F be a final spanning out-forest of a digraph D. We assume that there is an arc vw from F_i to F_j. Then*

(i) If $k_1 \leqslant i < j-k_2$, then D contains a $P(k_1,k_2)$.

(ii) If $k_1 < j \leqslant i-k_2$, then D contains a $P(k_1,k_2)$.

Proof. (i) Let P_{k_2} be the directed path of F which starts at F_{j-k_2} and ends at w and P_{k_1-1} be the directed path in F starting at $F_{i-(k_1-1)}$ and ending at v. Then $P_{k_1-1} \cup vw \cup P_{k_2}$ is a $P(k_1,k_2)$.

(ii) Let P_{k_2-1} be the directed path in F which starts at F_{i-k_2+1} and ends at v. Let P_{k_1} be the directed path in F starting at F_{j-k_1} and ending at w. Then $P_{k_1} \cup P_{k_2-1} \cup vw$ is a $P(k_1,k_2)$. □

Proof of Theorem 3.31. Let F be a final spanning out-forest of D. Colour the levels $F_1,\dots,F_{k_1}$ of F with colours $1,\dots,k_1$. Then colour the level F_i, where $i > k_1$, with colour $j \in \{k_1+1,\dots,k_1+k_2+1\}$ such that $j \equiv i \bmod k_2+1$. Since this is not a proper colouring, there exists an arc which satisfies the hypothesis of Lemma 3.32. □

3.3.3 Antidirected trees in digraphs

An interesting special case is the one of antidirected trees. Burr [18] proved that every antidirected tree of order k is contained in every digraph D with at least $4(k-1)v(D)$ arcs.

Theorem 3.33 (Burr [18])**.** *Let D be a digraph. If $a(D) > (4k-8)v(D)$, then D contains every antidirected tree of order k.*

Proof. Let D be a digraph of order n with more than $(4k-8)n$. Let (X,Y) be a bipartition of D that maximizes the number of arcs between X and Y. It is well-known that there are at least $a(D)/2$ such arcs. Without loss of generality, there are more arcs from X to Y than arcs from Y to X. Hence, there are more than $(k-2)n$ arcs from X to Y. Now remove iteratively from X all the vertices with outdegree at most $k-2$ and from Y all the vertices with indegree at most $k-2$. This process terminates on a non-empty bipartite digraph with bipartition (X',Y') in which every vertex of X' has outdegree at least $k-1$ and every vertex of Y' has indegree at least $k-1$. This certainly contains every antidirected tree of order k. $\square$

Remark 3.34. The condition of the tree to be antidirected is essential in Theorem 3.33. Indeed a bipartite digraph D with bipartition (X,Y) and all arcs from X to Y only contains antidirected tree while it can have up to $v(D)^2/4$ arcs.

Theorem 3.33 implies trivially that every antidirected tree of order k is $(8k-7)$-universal. Indeed every $(8k-7)$-critical digraph D has minimum degree at least $8k-8$ by Proposition 1.4 and thus has at least $4(k-1)v(D)$ arcs. This result has been improved by Addario-Berry et al. [2].They proved that every antidirected tree of order k is $(5k-\frac{17}{2})$-universal.

Since every antidirected tree of order k is contained in every digraph D with at least $4(k-1)v(D)$ arcs, one may ask what is the smallest function α_k such that every digraph D with more than $\alpha_k v(D)$ arcs contains every antidirected tree of order k. The above assertion shows that $\alpha_k \leqslant 4k-4$. Addario-Berry et al. [2] conjectured that $\alpha_k = k-2$.

Conjecture 3.35 (Addario-Berry et al. [2])**.** Let D be a digraph. If $a(D) > (k-2)v(D)$, then D contains every antidirected tree of order k.

Note that this conjecture would be tight, since the oriented tree of order k with a vertex dominating the $k-1$ others is not contained in any digraph in which every vertex has outdegree $k-2$. It is also tight since the complete symmetric digraph on $k-1$ vertices $\vec{K}_{k-1}$ has $(k-2)(k-1)$ arcs but does trivially not contain any oriented tree of order k.

Conjecture 3.35 would imply Burr's conjecture (Conjecture 3.20) for antidirected trees. Indeed every $(2k-2)$-critical digraph D has minimum degree at least $2k-3$ and thus has at least $\frac{2k-3}{2}v(D) > (k-2)v(D)$ arcs. Note that since a critical digraph is an oriented graph, it suffices to prove Conjecture 3.35 for oriented graphs to prove Burr's conjecture.

On the opposite, the well-known Erdős-Sós conjecture, reported in a paper of Erdős [30], is equivalent to Conjecture 3.35 for symmetric digraphs.

Conjecture 3.36 (Erdős and Sós, 1963)**.** Let G be a graph. If $e(G) > \frac{1}{2}(k-2)v(G)$, then G contains every tree of order k .

Indeed, consider a graph G and its corresponding symmetric digraph D (the digraph obtained from G by replacing each edge uv by the two arcs uv and vu). G has more than $\frac{1}{2}(k-2)v(G)$ edges if and only if $a(D) > (k-2)v(D)$. Let T be a tree and $\vec{T}$ one of its (two) antidirected orientations. It is simple matter to check that G contains T if and only if D contains $\vec{T}$.

The Erdős-Sós conjecture has been proved in particular cases: when the graph has no C_4 in [81]; and for trees with diameter at most four [66]. Ajtai-Komlos-Simonovits-Szemeredi announced that they showed that the conjecture holds for all sufficiently large k, using the Regularity Lemma, but the paper is not written yet.

3.4 Oriented cycles digraphs

3.4.1 Oriented cycles in tournaments

By Theorem 1.2, the oriented cycles are not universal. Moreover non-strong tournaments have clearly no hamiltonian directed cycles. Camion [20] showed all other tournaments do.

Theorem 3.37 (Camion [20])**.** *Every strong tournament has a hamiltonian directed cycle.*

Moon [69] strengthened this theorem by showing that every strong tournament is *vertex pancyclic*: each vertex lies in a directed cycle of every length ℓ ($3 \leqslant \ell \leqslant v(T)$).

Theorem 3.38 (Moon [69])**.** *Every strong tournament T is vertex-pancyclic.*

Proof. Let T be a strong tournament and v_1 a vertex of T. Let us prove by induction on $k \geqslant 3$ that v_1 is in a directed cycle of length k.

There is an arc v_2v_3 with $v_2 \in N^+(v_1)$ and $v_3 \in N^-(v_1)$ otherwise $N^-(v_1) \to N^+(v_1) \cup \{v_1\}$ would be a reduction of T. Hence $v_1v_2v_3$ is a directed 3-cycle..

Suppose now that v_1 is in a directed p-cycle $C = (v_1, v_2, \dots, v_, v_1)$. We shall prove that if $p < v(T)$ then v_1 is in a directed $(p+1)$-cycle. Let $S = V(D) \setminus V(C)$. Suppose that there is a vertex x of S that dominates a vertex of C and is dominated by a vertex of C. Then there is an arc $uv \in A(C)$ such that $u \to x$ and $x \to v$. Thus $xv \cup C[v,u], \cup ux$ is a directed $(p+1)$-cycle. Suppose now that there is no such vertex. Then for every vertex in S either $x \to C$ or $C \to x$. Let S^+ (resp. S^-) be the set of vertices x of S such that $C \to x$ (resp. $x \to C$). As T is strong there is an arc xy with $x \in S^+$ and $y \in S^-$. Hence $(v_1, x, y, v_3, \dots, v_p, v_1)$ is a directed $(p+1)$-cycle. $\square$

As we did for paths, one can seek arbitrary orientations of cycles. The existence of Grünbaum's exceptions implies the existence of tournaments that do not contain certain hamiltonian cycles. Indeed the tournament of Grünbaum's exceptions (see Figure 3) do not contain the cycle obtained from a hamiltonian antidirected path by adding an edge from its terminal vertex to its initial vertex. Moreover, the tournaments of order $n \in \{4, 6, 8\}$ that have a subtournament on $n-1$ vertices isomorphic to one of Grünbaum's tournaments do not contain a hamiltonian antidirected cycle.

However, similarly to oriented paths Rosenfeld [78] conjectured that there is an integer $N > 8$ such that every tournament of order $n \geqslant N$ contains every non-directed cycle of order n. This was settled by Thomason [89] for tournaments of order $n \geqslant 2^{128}$. While Thomason made no attempt to sharpen this bound, he indicated that it should be true for tournaments of order at least 9.

Conjecture 3.39 (Rosenfeld–Thomason)**.** Every tournament of order $n \geqslant 9$ contains every non-directed cycle of order n.

Havet [53] improved Thomason's results by showing that this conjecture is true for $n \geqslant 68$. The proof in based on different lemma which ensures the existence of a directed cycle in a tournament if it has a long block or not compared to its connectivity. In particular, the Conjecture 3.39 is true if the tournament is either reducible (see Exercice 3.17), or 8-strong [53]. It also true if the tournament is either 5-strong and of order at least 43 or 4-strong and of order at least 65.

Better results are also known for particular types of directed cycles. Conjecture 3.39 has been proved for cycles with a block of length $n-1$ by Grünbaum [42], for antidirected cycles (in which consecutives arcs have opposite senses) by Thomassen [90] ($n \geqslant 50$), Rosenfeld [78] ($n \geqslant 28$) and Petrović [73] ($n \geqslant 16$), and for cycles with just two blocks by Benhocine and Wojda [8].

3.4.2 Long directed cycles in digraphs

Furthermore, acyclic digraphs do not contain any directed cycles. However, Bondy showed that under the hypothesis of strong connectivity every digraph contains a long directed cycle.

Theorem 3.40 (Bondy [14])**.** *Every strong digraph D has a directed cycle of length at least $\chi(D)$.*

In order to prove this theorem, we need the notion of cyclic order. Let $D=(V,A)$ be a digraph. By a *cyclic order* of D we mean a cyclic order $O=(v_1,v_2,\ldots,v_n,v_1)$ of its vertex set V. Given such an order O, each directed cycle of D can be thought of as winding around O a certain number of times. In order to make this notion precise, we define the *length* of an arc (v_i,v_j) of D (with respect to O) to be $j-i$ if $i<j$ and $n+j-i$ if $i>j$. Informally, the length of an arc is just the length of the segment of O 'jumped' by the arc. If C is a directed cycle of D, the sum of the lengths of its arcs is a certain multiple of n. This multiple is called the *index* of C (with respect to O), and denoted $i_O(C)$. A directed cycle of index 1 is called a *simple cycle*. If every arc lies in such a cycle, the cyclic order is *coherent*.

Bessy and Thomassé [11] proved that every strong digraph has a coherent cyclic order. They then used such a cyclic order to prove Gallai's conjecture that every strong digraph D is spanned by the union of $\alpha(D)$ directed cycles. Camion's Theorem (3.37) derives also easily from the existence of a coherent cyclic order (See Exercise 3.18.). We shall now prove a slighty more general result in order to prove Theorem 3.40.

Lemma 3.41 (Bessy and Thomassé [11])**.** *Let D be a strong digraph. For every directed cycle C, there is coherent cyclic order of D for which C is simple.*

Proof. Consider a cyclic order O with respect to which C is simple that minimizes the *total index* of D, that is the sum of the indices over all directed cycles of D. Suppose for a contradiction that O is not coherent. Then there is an arc a which is in no simple directed cycles. Without loss of generality, $O = (v_1, v_2, \dots, v_n, v_1)$ and $a = v_k v_1$ for some k. Assume moreover that O has been chosen in order to minimize k. Let ℓ be the largest integer (smaller than k) such that there exists a directed (v_1, v_ℓ)-path with all vertices in $\{v_1, \dots, v_{k-1}\}$. Necessarily, v_ℓ has no outneighbours in $\{v_{\ell+1}, \dots, v_k\}$. If $\ell \neq 1$, then by the minimality of k, v_k has no inneighbours in $\{v_{\ell+1}, \dots, v_k\}$. In particular, the cyclic order $(v_1, \dots, v_{\ell-1}, v_{\ell+1}, \dots, v_k, v_\ell, v_{k+1}, \dots, v_n)$ has the same total index as O and contradicts the minimality of k. Thus $\ell = 1$, and by the minimality of k, there is no inneighbour of v_1 in $\{v_2, \dots, v_{k-1}\}$. Now consider the cyclic order $O' = (v_2, \dots, v_k, v_1, v_{k+1}, \dots, v_n)$. Every directed cycle C of D satisfies $i_{O'}(C) \leqslant i_O(C)$, and the inequality is strict if a is an arc of C. This contradicts the minimality of the total index with respect O. □

Proof of Theorem 3.40. Let $C = (u_1, \dots, u_k)$ be a longest directed cycle in D. By Lemma 3.41, there is a coherent cyclic order $O = (v_1, \dots, v_n, v_1)$ of D in which C is simple. Without loss of generality, there exists $1 \leqslant i_1 < i_2 < \cdots < i_k = n$ such that $u_j = v_{i_j}$ for all $1 \leqslant j \leqslant k$. For $1 \leqslant j \leqslant k$, let us define $I_j = v_{i_{j-1}} + 1, \dots, v_{i_j}$. An arc $v_l v_p$ is *bad* (with respect to O) if $l < p$ and l and p belong to the same I_j.

We assume that O has been chosen so that the number of bad arcs is minimized. We shall prove that there is no bad arcs. Since O is coherent, this will imply that every I_j is a stable set, and so $\chi(D) \leqslant k = v(C)$.

Suppose for a contradiction that there is a bad arc xy. Without loss of generality, we may assume that $x, y \in I_1$. Let D' be the subdigraph of D whose arc set is the set of bad arcs and arcs uv such that $u \in I_j$ and $v \in I_{j+1}$ for some $1 \leqslant j \leqslant k$ (in this proof all indices j are modulo k). Observe that D' contains C. Moreover, there is a directed (y,x)-path, for such a directed path would be of length at least k and such its union with (x,y) would give a directed cycle longer than C. Hence, in D', either there is no directed (C,x)-path, or there is no directed (y,C)-path. By directional symmetry, we may assume that the later case holds. Let Y be the set of vertices z such that there exists a directed (y,z)-path in D'. Set $K_j = I_j \cap Y$ and $L_j = I_j \setminus Y$, both with the ordering induced by O. By construction, for all $1 \leqslant j$, the last vertex of L_j is u_j and there is no arc from K_j to L_{j+1}. Moreover, because O is coherent, there is no arc from L_{j+1} to K_j. Now the cyclic order $O' = K_k L_1 K_1 L_2 K_2 \dots K_{k-1} L_k$ is coherent and C is simple with respect to it. Moreover every bad arc with respect to O' is also bad with respect to O. But the arc xy is not bad with respect to O'. This contradicts that O minimizes the number of bad arcs. □

We now prove a generalization of Bondy's theorem.

Let k be a positive integer and D be a digraph. A directed directed cycle C of D is called *k-good* if $v(C) \geqslant k$ and $\chi(D\langle V(C)\rangle) \leqslant k$. Note that Bondy's Theorem states that every strong digraph D has a $\chi(D)$-good directed cycle and that a shortest directed cycle is induced and thus is 3-good.

Theorem 3.42 (Addario-Berry, Havet and Thomassé [1]). *Let D be a strong digraph and k be in $\{3, \dots, \chi(D)\}$. Then D has a k-good directed cycle.*

Proof. By Theorem 3.40, there exists a directed cycle with length at least $\chi(D)$, implying the theorem for the value $k = \chi(D)$. Suppose $3 \leqslant k < \chi(D)$, in particular $\chi(D) > 3$. Let us now consider a shortest directed cycle C with length at least k. We claim that C is k-good. Suppose for contradiction that $\chi(D\langle V(C)\rangle) \geqslant k+1$. We may assume by induction on the number of vertices that $D = D\langle V(C)\rangle$. Furthermore, if D contains a directed 2-cycle, we can remove one of its arcs, in such a way that $\chi(D)$ and the directed cycle C are unchanged. Thus, we can assume that D has no directed 2-cycle, has a hamiltonian directed cycle C of length at least k, has chromatic number greater than k, and that every directed cycle of length at least k is hamiltonian. Our goal is to reach a contradiction.

We claim that every vertex u has indegree at most $k-2$ in D. Indeed, if $v_1, \dots, v_{k-1}$ were inneighbours of u, listed in such a way that $v_1, \dots, v_{k-1}, u$ appear in this order along C, the directed cycle obtained by shortcutting C through the arc $v_{k-2}u$ would have length at least k since the outneighbour of u in C is not an inneighbour of u. This contradicts the minimality of C. The same argument gives that every vertex has outdegree at most $k-2$ in D.

Let us then consider $H_1, \dots, H_r$, a handle decomposition of D with minimum number of trivial handles. Free to enumerate first the nontrivial handles, we can assume that $H_1, \dots, H_p$ are not trivial and $H_{p+1}, \dots, H_r$ are arcs. Let $D' := H_1 \cup \dots \cup H_p$. Clearly D' is a strong spanning subgraph of D. Observe that since $\chi(D) > 3$, D is not an induced directed cycle, so in particular $p > 1$.

We denote by $x_1, \dots, x_q$ the handle H_p minus its endvertices.

If $q = 1$, the digraph $D' - x_1$ is strong, and therefore $D - x_1$ is also strongly connected. Moreover its chromatic number is at least k. Thus by Bondy's theorem, there exists a directed cycle of length at least k in $D - x_1$. This directed cycle is not hamiltonian in D, a contradiction.

If $q = 2$, note that x_2 is the unique outneighbour of x_1 in D, otherwise we would make two non-trivial handles out of H_p, contradicting the maximality of the number of non-trivial handles. Similarly, x_1 is the unique inneighbour of x_2. Since the outdegree and the indegree of every vertex is at most $k-2$, both x_1 and x_2 have degree at most $k-1$ in the underlying graph of D. Since $\chi(D) > k$, it follows that $\chi(D - \{x_1, x_2\}) > k$. Since $D - \{x_1, x_2\}$ is strong, it contains, by Bondy's theorem, a directed cycle with length at least k, contradicting the minimality of C.

Hence, we may assume $q > 2$. For every $i = 1, \dots, q-1$, by the maximality of p, the unique arc in D leaving $\{x_1, \dots, x_i\}$ is $x_i x_{i+1}$ (otherwise we would make two non-trivial handles out of H_p). Similarly, for every $j = 2, \dots, q$, the unique arc in D entering $\{x_j, \dots, x_q\}$ is $x_{j-1}x_j$. In particular, as for $q = 2$, x_1 has outdegree 1 in D and x_q has indegree 1 in D.

Another consequence is that the underlying graph of $D \setminus \{x_1, x_q\}$ has two connected components $D_1 := D \setminus \{x_1, x_2, \dots, x_q\}$ and $D_2 := \{x_2, \dots, x_{q-1}\}$. Since the degrees of x_1 and x_q in the underlying graph of D are at most $k-1$ and D is at least $(k+1)$-chromatic, it follows that $\chi(D_1)$ or $\chi(D_2)$ is at least $(k+1)$-chromatic. Each vertex has indegree at most $k-2$ in D and $d^+_{D_2}(x_i) \leqslant 1$ for $2 \leqslant i \leqslant q-1$, so $\Delta(D_2) \leqslant k-1$ and $\chi(D_2) \leqslant k$. Hence D_1 is at least $(k+1)$-chromatic and strong. Thus by Bondy's Theorem, D_1 contains a directed cycle of length at least k but shorter than C. This is a contradiction. □

The existence of good directed cycles directly gives the existence of paths with two blocks in strong digraphs.

Lemma 3.43 (Addario-Berry, Havet and Thomassé [1]). *Let $k_1+k_2=k-1$ and D be a k-chromatic strong digraph. If D contains a (k_2+1)-good directed cycle then D contains a $P(k_1,k_2)$.*

Proof. Suppose C is a (k_2+1)-good directed cycle. Since $\chi(D\langle V(C)\rangle)\leqslant k_2+1$, the chromatic number of the (strong) contracted digraph D/C is at least k_1+1. Thus by Bondy's Theorem, D/C has a directed cycle of length at least k_1+1, and in particular the vertex C is the end of a path P of length k_1 in D/C. Finally $P\cup C$ contains a $P(k_1,k_2)$. □

Corollary 3.44 (Addario-Berry, Havet and Thomassé [1]). *Let $k_1+k_2=k-1\geqslant 3$ and D be a k-chromatic strong digraph. Then D contains a $P(k_1,k_2)$.*

Proof. Since $P(k_1,k_2)$ and $P(k_2,k_1)$ are isomorphic, we may assume that $k_2\geqslant 2$. By Lemma 3.42, D has an (k_2+1)-good directed cycle, and thus contains a $P(k_1,k_2)$ according to Lemma 3.43. □

A natural question is to ask for oriented cycles with two blocks instead of paths. As pointed out by Gyárfás and Thomassen, this does not extend to k-chromatic digraphs. Consider for this the following inductive construction: Let D_1 be the singleton digraph. Then, D_{i+1} is constructed starting with i disjoint copies $C_1,\dots,C_i$ of D_i and adding, for every set X of i vertices, one in each C_i, a vertex dominated exactly by X. By construction, the chromatic number of D_i is exactly i and there are no oriented cycle with two blocks.

However the digraphs D_i are not strong and it is easy to see that every strong digraph which is not a directed cycle contains two vertices x and y linked by two independent directed paths (i. e. having only x and y in common). We do not know if the strong connectivity condition ensures the existence of two vertices linked by two “long” independent paths.

Problem 3.45 (Addario-Berry, Havet and Thomassé [1]). Let D be a k-chromatic strong digraph ($k\geqslant 4$) and k_1, k_2 be positive integers such that $k_1+k_2=k$.
Does there exist two vertices of D which are linked by two independent directed paths P_1 and P_2 of length at least k_1 and k_2 respectively?
In other words, does there exists an oriented cycle with two blocks such that one block has length at least k_1 and the other one length at least k_2?

This problem may be seen as an extension of Bondy's theorem which proves this statement for directed cycles ($k_2=0$).

3.5 Exercises

Exercise 3.1. Give an elementaty proof that every tournament has a hamiltonian directed path.

Exercise 3.2. Erdős–Szekeres Theorem

1) Let D be a digraph with $\chi(D) \geqslant kl+1$, and let f be a real-valued function defined on $V(D)$. Show that D contains either a directed path $(u_0, u_1, \dots, u_k)$ with $f(u_0) \leqslant f(u_1) \leqslant \cdots \leqslant f(u_k)$ or a directed path $(v_0, v_1, \dots, v_l)$ with $f(v_0) > f(v_1) > \cdots > f(v_l)$.

(Chvátal and Komlós [23])

2) Deduce that any sequence of $kl+1$ distinct integers contains either an increasing subsequence of $k+1$ terms or a decreasing sequence of $l+1$ terms.

(Erdős and Szekeres [32])

Exercise 3.3. Let G be a graph that is not a forest and with chromatic number at most k for some integer $k \geqslant 2$. Show an acyclic orientation D of G such that $r(D) \leqslant k-1$.

Exercise 3.4. For a path partition $\mathcal{P}$, we denote the sets of initial and terminal vertices of its constituent paths by $i(\mathcal{P})$ and $t(\mathcal{P})$, respectively.

1) Show by induction on $v(D)$, that if $\mathcal{P}$ is a path partition of a digraph D such that no stable set of D is orthogonal to $\mathcal{P}$, then there is a path partition $\mathcal{Q}$ of D such that $|\mathcal{Q}| = |\mathcal{P}| - 1$, $i(\mathcal{Q}) \subset i(\mathcal{P})$, and $t(\mathcal{Q}) \subset t(\mathcal{P})$.

2) Deduce Theorem (3.11)

Exercise 3.5. Deduce Linial's Conjecture (3.12) from Berge's Path Partition Conjecture (3.13).

Exercise 3.6.

1) Show that a bipartite graph with average degree $2k$ or more contains a path of length $2k+1$.

(A. Gyarfás and J. Lehel [44])

2) Deduce that every antidirected path of length k is $4k$-universal.

Exercise 3.7. 1) Find a tournament on five vertices which contains no antidirected cycle.

2) Show that every 8-chromatic digraph contains an antidirected cycle.

(D. Grant, F. Jaeger, and C. Payan)

Exercise 3.8. Let D be an (acyclic) oriented graph with n vertices and m arcs. Show that there exists a tournament of order $2^{\frac{m}{n}}$ that contains no D.

Exercise 3.9. Verify the properties (M1) and (M2) of median orders of tournaments.

Exercise 3.10. Let D be a digraph and let $v_1, \dots, v_n$ be a median order D. Let D' be the spanning subdigraph of D with arc set $\{v_iv_j \in A(D) \mid i < j\}$.

1) We colour D by assigning to vertex v the colour $c(v)$, where $c(v)$ is the number of vertices of a longest increasing directed path in D' starting at v. Show that this colouring is proper.

2) Deduce Gallai–Roy Theorem.

Exercise 3.11. Show that every oriented tree on $k \geqslant 2$ vertices is $(4k-4)$-unavoidable.
(F. Havet and S. Thomassé [55])

Exercise 3.12. Let A be a rooted oriented tree. If the root is a source, we say that A is *well-rooted*. A (x,y) of A is a *forward arc* if x is in the oriented path between the root to y in A, and a *backward arc* otherwise. We denote the subdigraph of A induced by its backward arcs by $B(A)$, the number of backward arcs by $b(A)$ and the number of components of $B(A)$ by $c(A)$.

1) Show that every well-rooted tree A is $(v(A)+2b(A)-2c(A))$-unavoidable.

2) Deduce that every oriented tree on $k \geqslant 2$ vertices is $(3k-3)$-unavoidable.

(A. El Sahili [27])

Exercise 3.13. Let $v_1, v_2, \ldots, v_n$ be a median order of a tournament T on an even number of vertices. Show that $(v_1, v_2, \ldots, v_n, v_1)$ is a hamiltonian directed cycle of T. (S. Thomassé)

Exercise 3.14. A *king* in a tournament is a vertex v from which every vertex is reachable by a directed path of length at most two. Show that every tournament T has a king by proceeding as follows.

Let $v_1, v_2, \ldots, v_n$ be a median order of T.

1) Suppose that v_j dominates v_i, where $i < j$. Show that there is an index k with $i < k < j$ such that v_i dominates v_k and v_k dominates v_j.

2) Deduce that v_1 is a king in T. (F. Havet and S. Thomassé [55])

Exercise 3.15. A *second outneighbour* of a vertex v in a digraph is a vertex whose distance from v is exactly two. Show that every tournament T has a vertex with at least as many second outneighbours as (first) outneighbours, by proceeding as follows.

Let $v_1, v_2, \ldots, v_n$ be a median order of a tournament T. Colour the outneighbours of v_n red, both v_n and those of its inneighbours which dominate every red vertex preceding them in the median order black, and the remaining inneighbours of v_n blue. (Note that every vertex of T is thereby coloured, because T is a tournament.)

1) Show that every blue vertex is a second outneighbour of v_n.

2) Consider the intervals of the median order into which it is subdivided by the black vertices. Using property $(M2)$, show that each such interval includes at least as many blue vertices as red vertices.

3) Deduce that v_n has at least as many second outneighbours as outneighbours.

(F. Havet and S. Thomassé [55])

(P. D. Seymour has conjectured that every oriented graph has a vertex with at least as many second outneighbours as outneighbours.)

Exercise 3.16. Let T be a tournament of order n and let $P = (x_1,\dots,x_n)$ be an oriented path. We denote by $p(P,T)$ the number of distinct P that T contains. Let k be the largest integer such that $x_{k+1} \to x_k$ (with $k = 0$ if P is the converse of a directed path). Let $\tilde{P}$ be the oriented path obtained from P by reversing the arc $x_{k+1}x_k$ and set $Q = (x_1,\dots,x_k)$ and $R = (x_{k+1},\dots,x_n)$.

1) Let $\mathcal{B}$ be the sets of bipartition (A,B) of $V(T)$ such that $|A| = k$ and $|B| = n-k$. Show that
$$p(P,T) + p(\tilde{P},T) = \sum_{(A,B)\in\mathcal{B}} p(Q,T\langle A\rangle)\cdot p(R,T\langle B\rangle).$$

2) If n is non-negative integer, we define $U(n)$ as the set of integers i such that $n = \sum_{i\in U(n)} 2^i$. If m and n are integers, we say that $m \preceq n$ if $U(m) \subseteq U(n)$.
Show that $p(P,T) + p(\tilde{P},T) \equiv \binom{n}{k}\cdot L \pmod 2$ with L the number of $\preceq$-linearly ordered subsets of $\{i < k \mid x_{i+1}x_i \text{ and } i \preceq k\}$.

3) Knowing a result of Lucas stating that $\binom{n}{k}$ is odd if and only if $k \preceq n$, prove by induction on n and k that the parity of $p(P,T)$ equals the the number of $\preceq$-linearly ordered subsets of $\{i < n \mid x_{i+1}x_i \text{ and } i \preceq n\}$.

4) Deduce that if $n = 2^p$, then T contains an odd number of paths isomorphic to P.

(R. Forcade [36])

Exercise 3.17. Let $n \geqslant 9$. Show that every non-directed cycle of order n is contained in every non-strong tournament of order n.

Exercise 3.18. Deduce Camion's Theorem (3.37) from Lemma 3.41.
(S. Bessy and S. Thomassé [11])

4 The chromatic polynomial and acyclic orientations

Originally, the chromatic polynomial was introduced by Birkhoff [12] to attack the Four colour conjecture. As such, it was defined for planar graphs, but was extended to all graphs about twenty years later. Since then, the chromatic polynomial has been a core topic of algebraic graph theory. Further, it exhibited links with other areas, including knot theory. It was generalized by Tutte [91], to what is now called the Tutte polynomial, which proved to be an important and fundamental object generalizing several graph polynomials, with applications in, e.g., topology, statistical physics and probability theory.

Essentially, the idea of Birkhoff was to count the number different colourings of a graph to obtain more insight on its chromatic number. Two colourings of a graph G are considered to be distinct if at least one vertex is assigned different colours in the two colourings. Let $P_G(k)$ be the number of distinct proper k-colourings of G.

In particular, $P_G(k) > 0$ if and only if G is k-colourable. For instance, if G is a triangle then $P_G(2) = 0$ while $P_G(4) = 24$.

The *chromatic polynomial* is defined as the unique interpolating polynomial of degree $v(G)$ through the points $(k, P_G(k))$ for $k = 0, 1, ..., v(G)$.

If G is the complete graph on n vertices, then $P_G(k) = k(k-1)(k-2)\cdots(k-n+1)$. On the contrary, if G is the empty graph on n vertices, then $P_G(k) = k^n$, as every vertex can be assigned each of the k colours, independently of the other choices.

There is a simple recursion formula for P_G. Consider a graph G, and let u and v be two non-adjacent vertices of G. The set of all k-colourings of G can be partitioned into two subsets: the set $\mathcal{C}_s$ of those colourings c with $c(u) = c(v)$, and its complement $\mathcal{C}_d$. Note that every colouring in C_d is also a colouring of $G \cup uv$. Conversely, every colouring of $G \cup uv$ (using k colours) is a colouring of G that belongs to C_d. Similarly, colourings in C_s one-to-one correspond to colourings of the graph $G/\{u,v\}$ (which is obtained from G by identifying u and v). Consequently, we infer that $P_G(k) = P_{G\cup uv}(k) + P_{G/\{u,v\}}(k)$ for every pair $\{u,v\}$ of non-adjacent vertices of G. In other words, for every integer k, every graph G and every edge e of G,

$$P_G(k) = P_{G\setminus e}(k) - P_{G/e}(k)\,. \tag{5}$$

By the definition, the value of the chromatic polynomial $P_G(x)$ at a positive integer k is the number of proper k-colourings of G. Surprisingly, evaluations of the chromatic polynomial at certain other special value of x have interesting interpretation. An example is the following theorem of Stanley [87], which is another link between the orientations and the colourings of a graph.

Theorem 4.1 (Stanley [87]). *For every graph G, the number of acyclic orientations of G is $(-1)^{v(G)}P_G(-1)$.*

Proof. For a graph G, let $\mathscr{O}(G)$ be the set of all acyclic orientations of G, and set $o(G) = |\mathscr{O}(G)|$. We prove that $o(G) = (-1)^{v(G)}P_G(-1)$ by induction on $e(G)$. The statement is true for the empty graph on n vertices: such a graph has exactly one acyclic orientation, and its chromatic polynomial is X^n.

Suppose now that the conclusion holds for graphs with less than $e(G)$ edges. First, note that every acyclic orientation of G yields an acyclic orientation of $G \setminus e$. Conversely, notice that an acyclic orientation of $G \setminus e$ cannot contain both a directed (u,v)-path and a directed (v,u)-path.

Let $\mathscr{O}_2 \subseteq \mathscr{O}(G\setminus e)$ be the set of those acyclic orientations of $G \setminus e$ with neither a path from u to v, nor a path from v to u. Set $\mathscr{O}_1 = \mathscr{O}(G\setminus e) \setminus \mathscr{O}_2$. Thus, every orientation in $\mathscr{O}_i$ can be extended in exacly i ways into an acyclic orientation of G. As a result, $|\mathscr{O}(G)| = |\mathscr{O}_1| + 2\cdot|\mathscr{O}_2|$.

By the induction hypothesis, $o(G\setminus e) = (-1)^{v(G)}P_{G\setminus e}$.

Now, observe that every acyclic orientation of G/e yields an acyclic orientation of $G \setminus e$ that belongs to $\mathscr{O}_2$, and, conversely, an acyclic orientation a of $G \setminus e$ yields an acyclic orientation of G/e if and only if $a \in \mathscr{O}_2$. Consequently, using the induction hypothesis, it follows that $|\mathscr{O}_2| = o(G/e) = (-1)^{v(G)-1}P_{G/e}(-1)$. The conclusion follows. $\square$

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