7. The regularity lemma for digraphs

Given a digraph G and a vertex $x \in V(G)$, the indegree $d_G^-(x)$ of x in G is the number of edges of G incident to x that are oriented towards x. Similarly the outdegree $d_G^+(x)$ of x in G is the number of edges of G incident to x that are oriented away from x. The minimum indegree $\delta^-(G)$ of G is the minimum value of $d_G^-(x)$ over all $x \in V(G)$. Similarly the minimum outdegree $\delta^+(G)$ of G is the minimum value of $d_G^+(x)$ over all $x \in V(G)$. The minimum semidegree $\delta^0(G)$ of G is the minimum of $\delta^-(G)$ and $\delta^+(G)$.

Given disjoint vertex sets A and B in a digraph G, we write $(A,B)_G$ for the oriented bipartite subgraph of G whose vertex classes are A and B and whose edges are all the edges from A to B in G. We say $(A,B)_G$ is *ε-regular and has density d* if the underlying (undirected) bipartite graph of $(A,B)_G$ is ε-regular and has density d. (Note that the ordering of the pair (A,B) is important here.) We say G is *(ε,d)-super-regular* if it is ε-regular and $\delta^+(a) \geq d|B|$ for all $a \in A$ and $d^-(b) \geq d|B|$ for all $b \in B$.

These definitions generalize naturally to non-bipartite (di-)graphs. In particular, we say that a digraph G on n vertices is *ε-regular* if there is a d so that $\frac{e(X,Y)}{|X||Y|} = d \pm \varepsilon$ for all (not necessarily disjoint) subsets X, Y of $V(G)$ of size at least εn. Here $e(X,Y)$ denotes the number of edges of G from X to Y. We say $(A,B)_G$ is *(ε,d)-super-regular* if it is ε-regular and $\delta^0(G) \geq dn$.

The Diregularity lemma is a variant of the Regularity lemma for digraphs (due to Alon and Shapira [1]). Its proof is similar to the undirected version. We will use the degree form of the Diregularity lemma which is derived from the standard version in the same manner as the undirected degree form.

Lemma 7.1 (Degree form of the Diregularity lemma)**.** *For every $\varepsilon \in (0,1)$ and every integer M' there are integers M and n_0 such that if G is a digraph on $n \geq n_0$ vertices and $d \in [0,1]$ is any real number, then there is a partition of the vertex set of G into $V_0, V_1, \dots, V_k$ and a spanning subdigraph G' of G such that the following holds:*

- *$M' \leq k \leq M$,*
- *$|V_0| \leq \varepsilon n$,*
- *$|V_1| = \dots = |V_k| =: m$,*
- *$d_{G'}^+(x) > d_G^+(x) - (d+\varepsilon)n$ for all vertices $x \in V(G)$,*
- *$d_{G'}^-(x) > d_G^-(x) - (d+\varepsilon)n$ for all vertices $x \in V(G)$,*
- *for all $i = 1, \dots, k$ the digraph $G'[V_i]$ is empty,*
- *for all $1 \leq i, j \leq k$ with $i \neq j$ the pair $(V_i, V_j)_{G'}$ is ε-regular and has density either 0 or density at least d.*

We call $V_1, \dots, V_k$ *clusters*, V_0 the *exceptional set* and the vertices in V_0 *exceptional vertices*. We refer to G' as the *pure digraph*. The last condition of the lemma says that all pairs of clusters are ε-regular in both directions (but possibly with different densities). The *reduced digraph R of G with parameters ε, d and M'* is the digraph whose vertices are $V_1, \dots, V_k$ and in which V_iV_j is an edge precisely when $(V_i, V_j)_{G'}$ is ε-regular and has density at least d.

8. Cycles of given length in oriented graphs

A digraph is an *oriented graph* if it is an orientation of a simple graph. A central problem in digraph theory is the Caccetta-Häggkvist conjecture [2]:

Conjecture 8.1. *An oriented graph on n vertices with minimum outdegree d contains a cycle of length at most $\lceil n/d \rceil$.*

A special case of Conjecture 8.1 that has attracted much interest is when $d = \lceil n/3 \rceil$. The following bound towards this case is due to Shen [8].

Theorem 8.2. *If G is any oriented graph on n vertices with $\delta^+(G) \geq 0.355n$ then G contains a directed triangle.*

We consider the natural and related question of which minimum semidegree forces cycles of length exactly $\ell \geq 4$ in an oriented graph. We will often refer to cycles of length ℓ as *ℓ-cycles*. We conjecture that the correct bounds are those given by the obvious extremal example: when we seek an ℓ-cycle, the extremal example is probably the blow-up of a k-cycle, where $k \geq 3$ is the smallest integer which is not a divisor of ℓ.

Conjecture 8.3 ([3])**.** *Let $\ell \geq 4$ be a positive integer and let k be the smallest integer that is greater than 2 and does not divide ℓ. Then there exists an integer $n_0 = n_0(\ell)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \lfloor n/k \rfloor + 1$ contains an ℓ-cycle.*

It is easy to see that the only values of k that can appear in Conjecture 8.3 are of the form $k = p^s$ with $k \geq 3$, where $p \geq 2$ is a prime and s a positive integer. The following result implies that Conjecture 8.3 is approximately true when $k = 4$ and ℓ is sufficiently large. More general results were obtained in [3, 6]

Theorem 8.4 ([3])**.** *Let $\ell \geq 42$ be a positive integer and let k be the smallest integer that is greater than 2 and does not divide ℓ. If $k = 4$ then for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\ell, \varepsilon)$ such that every oriented graph G on $n \geq n_0$ vertices with $\delta^0(G) \geq n/k + \varepsilon n$ contains an ℓ-cycle.*

Before we begin the proof of this theorem we state the following useful fact.

Fact 8.5. *If G is an oriented graph on n vertices then the maximum size of an independent set is at most $n - 2\delta^0(G)$.* □

We will prove Theorem 8.4 using the following two lemmas. The first lemma implies that if we allow ourselves a linear 'error term' in the degree conditions then instead of finding an ℓ-cycle, it suffices to look for a closed walk of length ℓ.

Lemma 8.6. *Let $\ell \geq 2$ be an integer. Suppose that $c > 0$ and there exists an integer n_0 such that every oriented graph H on $n \geq n_0$ vertices with $\delta^0(H) \geq cn$ contains a closed walk W of length ℓ. Then for each $\epsilon > 0$ there exists $n_1 = n_1(\epsilon, \ell, n_0)$ such that if G is an oriented graph on $n \geq n_1$ vertices with $\delta^0(G) \geq (c+\epsilon)n$ then G contains an ℓ-cycle.*

The proof of Lemma 8.6 is a standard application of the Regularity lemma for digraphs.

Sketch of proof of Lemma 8.6. Apply the degree form of the directed Regularity lemma to G to obtain a partition of $V(G)$ into clusters and a reduced digraph R'. So the vertices of R' are the clusters and there is a directed edge from A to B in R' if the bipartite subdigraph of G consisting of the edges from A to B is ε'-regular and has density at least d, where $\varepsilon' \ll d \ll \varepsilon$. One can show that R' almost inherits the minimum semidegree of G, i.e. $\delta^0(R') \geq (c+\varepsilon/2)|R'|$. However, R' need not be oriented. But for every double edge of R' one can delete one of the two edges randomly (with suitable probability) in order to obtain an oriented spanning subgraph R of R' which still satisfies $\delta^0(R) \geq c|R|$ (see [4, Lemma 8] for a proof). Applying our assumption with $H := R$ gives a closed walk of length ℓ in R. Since n_1 is large compared to ℓ, this also holds for size of the clusters. So we can apply the Key lemma to find an ℓ-cycle in G. □

Lemma 8.7. *Let G be an oriented graph on n vertices. If $\delta^0(G) \geq n/4$ then either the diameter of G is at most 6 or G contains a 3-cycle.*

Proof. Consider $x \in V(G)$ and define $X_1 := N^+(x)$ and $X_{i+1} := N^+(X_i) \cup X_i$ for $i \geq 1$. If there exists an i with $\delta^+(G[X_i]) > 3|X_i|/8$ then $G[X_i]$ contains a 3-cycle by Theorem 8.2. So assume not. Then there exists a vertex $x_i \in X_i$ with $|N^+(x_i) \cap X_i| \leq 3|X_i|/8$. Hence

$$|X_{i+1}| \geq |X_i| + (\delta^0(G) - 3|X_i|/8) \geq 5|X_i|/8 + n/4.$$

In particular $|X_2| \geq 13n/32$ and $|X_3| \geq 65n/256 + n/4 = 129n/256 > n/2$. Similarly, for any vertex $y \neq x$ we have that $|\{v \in V(G) : dist(v, y) \leq 3\}| > n/2$, and thus there exists an x-y path of length at most 6, which completes the proof. □

Proof of Theorem 8.4. By Lemma 8.6 it suffices to show that every sufficiently large oriented graph H with $\delta^0(H) \geq |H|/4+1$ contains a closed walk of length ℓ. If H has a 3-cycle then it contains such a walk since 3 divides ℓ by definition of k. Thus we may assume that H has no 3-cycle. Fact 8.5 implies that the maximum size of an independent set is smaller than the neighbourhood $N_H(v)$ of any vertex v. Thus H contains some orientation of a triangle. By assumption this is not a 3-cycle, and so it must be transitive, i.e. the triangle consists of vertices x, y, z and edges xz, xy, zy.

Since $H - z$ has no 3-cycle, Lemma 8.7 implies that $H - z$ contains a y-x path P of length $t \leq 6$. This gives us 2 cycles $C_1 := yPxy$ and $C_2 := yPxzy$ of lengths $t+1$ and $t+2$ respectively. Write ℓ as $\ell = a(t+1) + r$ with $0 \leq r \leq t \leq 6$. We can wind r times around C_2 and $(a-r)$ times around C_1 to find a closed walk of length ℓ in H provided that $r \leq a$. But the latter holds as $a = \lfloor \ell/(t+1) \rfloor \geq 6$. □

Probably the use of the Regularity lemma is heavy-handed in this instance – it would be interesting to obtain a proof which does not use it.

9. Outexpanders and Hamilton cycles

Roughly speaking, a graph is an expander if for every set S of vertices the neighbourhood $N(S)$ of S is significantly larger than S itself. A number of papers have recently demonstrated that there is a remarkably close connection between Hamiltonicity and expansion. The following notion of robustly expanding (dense) digraphs was introduced in [7].

Let $0 < \nu \leq \tau < 1$. Given any digraph G on n vertices and $S \subseteq V(G)$, the *ν-robust outneighbourhood $RN^+_{\nu,G}(S)$ of S* is the set of all those vertices x of G which have at least νn inneighbours in S. G is called a *robust (ν, τ)-outexpander* if $|RN^+_{\nu,G}(S)| \geq |S| + \nu n$ for all $S \subseteq V(G)$ with $\tau n < |S| < (1-\tau)n$. As the name suggests, this notion has the advantage that it is preserved even if we delete some vertices and edges from G.

Theorem 9.1 (Kühn, Osthus and Treglown [7])**.** *Let n_0 be a positive integer and ν, τ, η be positive constants such that $1/n_0 \ll \nu \leq \tau \ll \eta < 1$. Let G be a digraph on $n \geq n_0$*

vertices with $\delta^0(G) \geq \eta n$ which is a robust (ν, τ)-outexpander. Then G contains a Hamilton cycle.

The result has a number of applications, e.g. an (approximate) analogue of Chvátal's and Pósa's theorems for digraphs, and an (approximate) analogue of Dirac's theorem for oriented graphs. It is also used as a tool in the proof of Kelly's Conjecture.

Theorem 9.2. *For every $\alpha > 0$ there exists an integer $N = N(\alpha)$ such that every oriented graph G of order $|G| \geq N$ with $\delta^0(G) \geq (3/8 + \alpha)|G|$ contains a Hamilton cycle.*

Theorem 9.3. *For every $\eta > 0$ there exists an integer $n_0 = n_0(\eta)$ such that the following holds. Suppose G is a digraph on $n \geq n_0$ vertices such that for all $k < \frac{n}{2}$*

(i) $d^+_k \geq k + \eta n$ *or* $d^-_{n-k-\eta n} \geq n - k$;
(ii) $d^-_k \geq k + \eta n$ *or* $d^+_{n-k-\eta n} \geq n - k$.

Then G contains a Hamilton cycle.

The original proof of Theorem 9.1 relied on the Blow-up Lemma. Below, we give a brief sketch of a proof which avoids any use of the Blow-up lemma.

The following result states that robust outexpansion is inherited by the reduced digraph.

Lemma 9.4. *Let M', n_0 be positive integers and let $\varepsilon, d, \eta, \nu, \tau$ be positive constants such that $1/n_0 \ll \varepsilon \ll d \ll \nu, \tau, \eta < 1$ and such that $M' \ll n_0$. Let G be a digraph on $n \geq n_0$ vertices with $\delta^0(G) \geq \eta n$ and such that G is a robust (ν, τ)-outexpander. Let R be the reduced digraph of G with parameters ε, d and M'. Then $\delta^0(R) \geq \eta|R|/2$ and R is a robust $(\nu/2, 2\tau)$-outexpander.*

Proof. Let G' be as in the degree form of the diregularity lemma, $k := |R|$, let $V_1, \dots, V_k$ be the clusters of G (i.e. the vertices of R) and V_0 the exceptional set. Let $m := |V_1| = \dots = |V_k|$. Then

$$\delta^0(R) \geq (\delta^0(G') - |V_0|)/m \geq (\delta^0(G) - (d + 2\varepsilon)n)/m \geq \eta k/2.$$

Consider any $S \subseteq V(R)$ with $2\tau k \leq |S| \leq (1 - 2\tau)k$. Let S' be the union of all the clusters belonging to S. Then $\tau n \leq |S'| \leq (1 - 2\tau)n$. Since $|N^-_{G'}(x) \cap S'| \geq |N^-_G(x) \cap S'| - (d + \varepsilon)n \geq \nu n/2$ for every $x \in RN^+_{\nu,G}(S')$ this implies that

$$|RN^+_{\nu/2,G'}(S')| \geq |RN^+_{\nu,G}(S')| \geq |S'| + \nu n \geq |S|m + \nu mk.$$

However, in G' every vertex $x \in RN^+_{\nu/2,G'}(S') \backslash V_0$ receives edges from vertices in at least $|N^-_{G'}(x) \cap S'|/m \geq (\nu n/2)/m \geq \nu k/2$ clusters $V_i \in S$. Thus by the final property of the partition in Lemma 7.1 the cluster V_j containing x is an outneighbour of each such V_i (in R). Hence $V_j \in RN^+_{\nu/2,R}(S)$. This in turn implies that

$$|RN^+_{\nu/2,R}(S)| \geq (|RN^+_{\nu/2,G'}(S')| - |V_0|)/m \geq |S| + \nu k/2,$$

as required. □

We also need the result that every super-regular digraph contains a Hamilton cycle.

Lemma 9.5. *Suppose that $1/n_0 \ll \varepsilon \ll d \ll 1$ and G is an (ε, d)-super-regular digraph on $n \geq n_0$ vertices. Then G contains a Hamilton cycle.*

Sketch proof of Lemma 9.5. We first prove that G contains a 1-factor. Consider the auxiliary bipartite graph whose vertex classes A and B are copies of $V(G)$ with an edge between $a \in A$ and $b \in B$ if there is an edge from a to b in G. One can show that this bipartite graph has a perfect matching (by Hall's marriage theorem), which in turn corresponds to a 1-factor in G.

It is now not hard to prove the lemma using the 'rotation-extension' technique: Choose a 1-factor of G. Now remove an edge of a cycle in this 1-factor and let P be the resulting path. If the final vertex of P has any outneighbours on another cycle C of the 1-factor, we can extend P into a longer path which includes the vertices of C (and similarly for the initial vertex of P). We repeat this as long as possible (and one can always ensure that the extension step can be carried out at least once). So we may assume that all outneighbours of the final vertex of P lie on P and similarly for the initial vertex of P. Together with the ε-regularity this can be used to find a cycle with the same vertex set as P. Eventually, we arrive at a Hamilton cycle. $\square$

Sketch proof of Theorem 9.1.[1] Choose ε, d to satisfy $1/n_0 \ll \varepsilon \ll d \ll \nu$. The first step is to apply the directed version of Szemerédi's regularity lemma to G. This gives us a partition of the vertices of G into clusters $V_1, \ldots, V_k$ and an exceptional set V_0 so that $|V_0| \leq \varepsilon n$ and all the clusters have size m. Now define the 'reduced' digraph R whose vertices are the clusters $V_1, \ldots, V_k$. Lemma 9.4 implies that R is still a $(\nu/2, 2\tau)$-outexpander (this is the point where we need the robustness of the expansion in G) with minimum semidegree at least $\eta k/2$. This in turn can be used to show that R has a 1-factor $\mathcal{F}$ (using the same auxiliary bipartite graph as in the proof of Lemma 9.5). By removing a small number of vertices from the clusters, we can also assume that the bipartite subgraphs spanned by successive clusters on each cycle of $\mathcal{F}$ are super-regular, i.e. have high minimum degree. For simplicity, assume that the cluster size is still m.

Moreover, since G is an expander, we can find a short path in G between clusters of different cycles of $\mathcal{F}$ and also between any pair of exceptional vertices. However, we need to choose such paths without affecting any of the useful structures that we have found so far. For this, we will consider paths which 'wind around' cycles in $\mathcal{F}$ before moving to another cycle. More precisely, a *shifted walk* from a cluster A to a cluster B is a walk $W(A, B)$ of the form

$$W(A, B) = X_1 C_1 X_1^- X_2 C_2 X_2^- \ldots X_t C_t X_t^- X_{t+1},$$

where $X_1 = A$, $X_{t+1} = B$, C_i is the cycle of $\mathcal{F}$ containing X_i, and for each $1 \leq i \leq t$, X_i^- is the predecessor of X_i on C_i and the edge $X_i^- X_{i+1}$ belongs to R. We say that W as above *traverses t cycles* (even if some C_i appears several times in W). We also say that the clusters $X_2, \ldots, X_{t+1}$ are the *entry clusters* (as this is where W 'enters' a cycle C_i) and the clusters $X_1^-, \ldots, X_t^-$ are the *exit clusters* of W. Note that

(i) for any cycle of $\mathcal{F}$, its clusters are visited the same number of times by $W(A, B) - B$.

Using the expansion of R, it is not hard to see that

[1] This proof follows a survey on digraph Hamilton cycles by D. Kühn and D. Osthus [5]

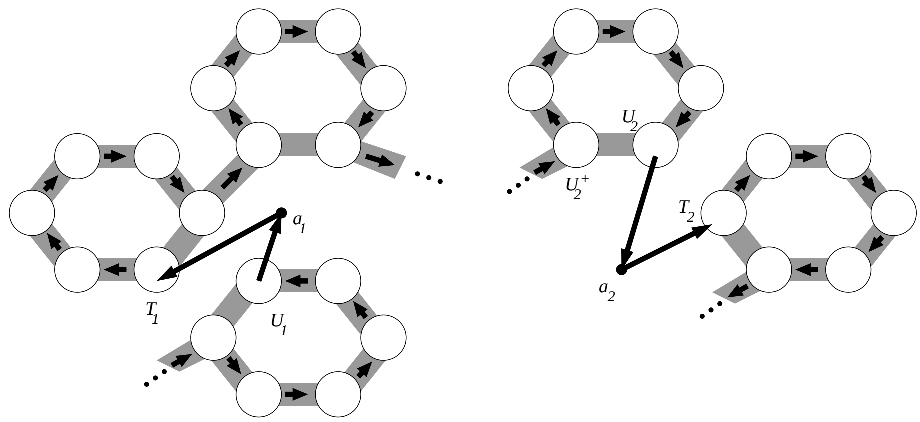

FIGURE 1. Constructing the walk W

(ii) for any clusters A and B there is a shifted walk from A to B which does not traverse too many cycles.

Indeed, the expansion property implies that the number of clusters one can reach by traversing t cycles is at least $t\nu k/2$ as long as this is significantly less than the total number k of clusters.

Now we will 'join up' the exceptional vertices using shifted walks. For this, write $V_0 = \{a_1, \dots, a_\ell\}$. For each exceptional vertex a_i choose a cluster T_i so that a_i has many outneighbours in T_i. Similarly choose a cluster U_i so that a_i has many inneighbours in U_i and so that

(iii) no cluster appears too often as a T_i or a U_i.

Given a cluster X, let X^- be the predecessor of X on the cycle of $\mathcal{F}$ which contains X and let X^+ be its successor. Form a 'walk' W on $V_0 \cup V(R)$ which starts at a_1, then moves to T_1, then follows a shifted walk from T_1 to U_2^+, then it winds around the entire cycle of $\mathcal{F}$ containing U_2^+ until it reaches U_2. Then W moves to a_2, then to a_3 using a shifted walk as above until it has visited all the exceptional vertices (see Figure 5). Proceeding similarly, we can ensure that W has the following properties:

(a) W is a closed walk which visits all of V_0 and all of $V(R)$.
(b) For any cycle of $\mathcal{F}$, its clusters are visited the same number of times by W.
(c) Every cluster appears at most $m/10$ times as an entry or exit cluster.

(b) follows from (i) and (c) follows from (ii) and (iii). The next step towards a Hamilton cycle would be to find a cycle C in G which corresponds to W (i.e. each occurrence of a cluster in W is replaced by a distinct vertex of G lying in this cluster). Unfortunately, the fact that V_0 may be much larger than the cluster size m implies that there may be clusters which are visited more than m times by W, which makes it impossible to find such a C. So we will apply a 'short-cutting' technique to W which avoids 'winding around' the cycles of $\mathcal{F}$ too often.

For this, we now fix edges in G corresponding to all those edges of W that do not lie within a cycle of $\mathcal{F}$. These edges of W are precisely the edges in W at the exceptional vertices as well as all the edges of the form AB where A is used as an exit cluster by W and B is used as an entrance cluster by W. For each edge a_iT_i at an exceptional vertex we choose an edge a_ix, where x is an outneighbour of a_i in T_i. We similarly choose an edge ya_i from U_i to a_i for each U_ia_i. We do this in such a way that all these

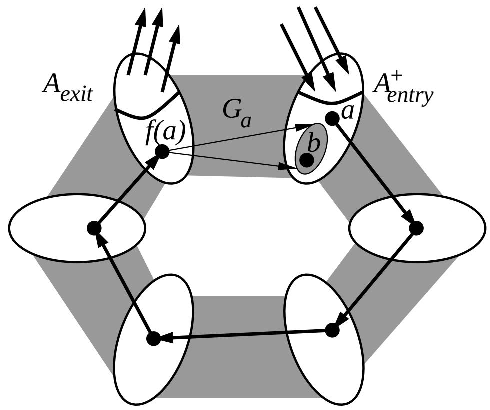

FIGURE 2. An illustration of the auxiliary digraph J, where there is an edge from a to b in J

edges are disjoint outside V_0. For each occurrence of AB in W, where A is used as an exit cluster by W and B is used as an entrance cluster, we choose an edge ab from A to B in G so that all these edges are disjoint from each other and from the edges chosen for the exceptional vertices (we use (c) here).

Given a cluster A, let A_{entry} be the set of all those vertices in A which are the final vertex of an edge of G fixed so far and let A_{exit} be the set of all those vertices in A which are the initial vertex of an edge of G fixed so far. So $A_{entry} \cap A_{exit} = \emptyset$. Let G_A be the bipartite graph whose vertex classes are $A \setminus A_{exit}$ and $A^+ \setminus A^+_{entry}$ and whose edges are all the edges from $A \setminus A_{exit}$ to $A^+ \setminus A^+_{entry}$ in G. Since W consists of shifted walks, it is easy to see that the vertex classes of G_A have equal size. Moreover, it is possible to carry out the previous steps in such a way that G_A is super-regular (here we use (c) again). This in turn means that G_A has a perfect matching M_A. These perfect matchings (for all clusters A) together with all the edges of G fixed so far form a 1-factor $\mathcal{C}$ of G. It remains to transform $\mathcal{C}$ into a Hamilton cycle.

Claim. For any cluster A, we can find a perfect matching M'_A in G_A so that if we replace M_A in $\mathcal{C}$ with M'_A, then all vertices of G_A will lie on a common cycle in the new 1-factor $\mathcal{C}$.

To prove this claim we proceed as follows. For every $a \in A^+ \setminus A^+_{entry}$, we move along the cycle C_a of $\mathcal{C}$ containing a (starting at a) and let $f(a)$ be the first vertex on C_a in $A \setminus A_{exit}$. Define an auxiliary digraph J on $A^+ \setminus A^+_{entry}$ such that $N^+_J(a) := N^+_{G_A}(f(a))$. So J is obtained by identifying each pair $(a, f(a))$ into one vertex with an edge from $(a, f(a))$ to $(b, f(b))$ if G_A has an edge from $f(a)$ to b (see Figure 6). Since G_A is super-regular, it follows that J is also super-regular. By Lemma 9.5, J has a Hamilton cycle, which clearly corresponds to a perfect matching M'_A in G_A with the desired property.

We now repeatedly apply the above claim to every cluster. Since $A_{entry} \cap A_{exit} = \emptyset$ for each cluster A, this ensures that all vertices which lie in clusters on the same cycle of $\mathcal{F}$ will lie on the same cycle of the new 1-factor $\mathcal{C}$. Since by (a) W visits all clusters, this in turn implies that all the non-exceptional vertices will lie in the same cycle of $\mathcal{C}$. Since the exceptional vertices form an independent set in $\mathcal{C}$, it follows that $\mathcal{C}$ is actually a Hamilton cycle. $\square$

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