Glauber dynamics for edge colorings of trees

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November 15, 2018

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- Start from an arbitrary coloring.
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Question

How long is 'long enough'? Polynomial (in the size of G)?

Motivations

- Generate random colorings of graphs.
- Approximation algorithms for counting problems.
- Motivation from statistical physics: Potts model, (Generalization of the Ising model).

Markov Chain

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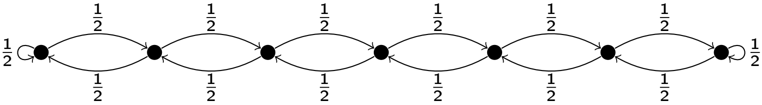

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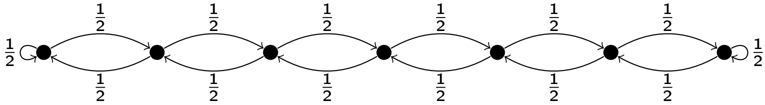

- $X_0 P^t$: distribution after t steps.
- Ω : set of states.

Markov Chain

- Irreducible: the graph is strongly connected.
- Aperiodic: "non-zero probability to stay in the same place".
- Total variation distance $\|\mu - \eta\|_{TV} = \sum_{x \in \Omega} |\mu(x) - \eta(x)|$

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- Total variation distance $\|\mu - \eta\|_{TV} = \sum_{x \in \Omega} |\mu(x) - \eta(x)|$
- Irreducible + Aperiodic $\Rightarrow$ Ergodic : There is a unique stationary distribution π.
- $\pi P = \pi$
- $\lim_{t \to \infty} X_0 P^t = \pi$
- Mixing time: $\tau = \min \left\{ t,\ \max_{X_0} \|X_0 P^t - \pi\|_{TV} \leq \frac{1}{2} \right\}$

Markov Chain

- $\Omega =$ set of all possible k-colorings.

$$P[\sigma \to \tau] = \begin{cases} \frac{1}{kn} & \text{if } \sigma \text{ and } \tau \text{ differ on only one vertex} \\ 0 & \text{otherwise} \end{cases}$$

- $P[\sigma \to \tau] = P[\tau \to \sigma]$
- The stationary distribution is uniform.
- The process is called **Glauber Dynamics**, noted $\mathcal{L}_{GD}$.

Known Results

If $k \geq \Delta + 2$, the process is ergodic.

[M. R. Jerrum, 1994], [E. Vigoda, 1999], [S. Chen, A. Moitra, 2018], [M. Delcourt, G. Perarnau, L. Postle, 2018]

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Class of graph	# of colors	Mixing time	Reference
General graphs	$k > 2\Delta$	$O(n \log n)$	[Jer94]
	$k > \frac{11}{6}\Delta$	$O(n^2 \log n)$	[Vig99]
	$(\frac{11}{6} - \varepsilon)\Delta$	$O(n^2)$	[CM18, DPP18]

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Class of graph	# of colors	Mixing time	Reference
Graphs with girth ≥ 7	1.489Δ	$O(n \log n)$	[DFHV13]
Planar Graphs	$\Omega(\frac{\Delta}{\log \Delta})$	$O(n^3 \log^9 n)$	[HVV07]
Trees	$k \geq 3$	$n^{O(1+\frac{\Delta}{k \log \Delta})}$	[LMP09]
Edge coloring complete tree	2Δ	$\mathrm{poly}(n)$	[Poo16]

[M. Dyer, A Frieze, T. P. Hayes, E. Vigoda, 2013], [T. P. Hayes, J. C. Vera, E. Vigoda, 2007], [B. Lucier, M. Molloy; Y. Peres, 2009], [C. Y. Poon, 2016]

Results

Theorem

Glauber dynamics for edge colorings of a tree with $\Delta + 1$ colors mixes in polynomial time.

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- The number of colors is tight.
- Proof for complete regular trees.

General methods

- Coupling

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- Comparison of Markov Chains:
  - given Markov chain $\mathcal{L}$ with mixing time τ,
  - define some modified dynamics $\mathcal{L}'$ with τ',
  - bound τ in terms of τ'.

Block dynamics

Definition

Let $V_1, \ldots, V_\ell$ be a partition of the vertices. Consider the process $\mathcal{L}_B$ where at each step:

- Select a block at random.
- Choose a new coloring of this block uniformly at random.

[Martinelli, 2000]

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Theorem ([Mar00])

If the Glauber Dynamics restricted to each block are ergodic, then:

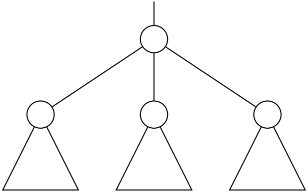
$$\tau(\mathcal{L}_{GD}) \leq \tau(\mathcal{L}_B) \cdot \max_i \tau(\mathcal{L}_{GD}|_{V_i})$$

[Martinelli, 2000]

Complete trees

Idea

Recursively decompose the tree using block dynamics.



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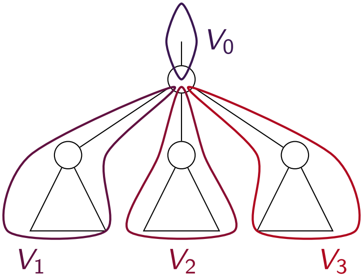

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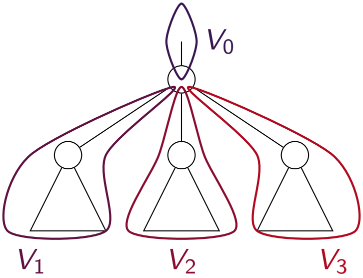

$$\tau(h) \leq \tau(h-1)\tau(\mathcal{L}_B)$$

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Lemma

The mixing time for the Glauber dynamics for edge coloring a star of size Δ is at most:

$$\tau^* \leq \mathrm{poly}(\Delta)$$

Conclusion

- Edge coloring for other graphs:
  - K_n, $K_{n,n}$,
  - chordal graphs, interval graphs...
  - graphs with bounded treewidth,
  - random d-regular graphs.
- Number of colours necessary for ergodicity of edge colorings.
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Thank You!