Achromatic number of signed graphs

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Supervised by Hervé Hocquard and Éric Sopena

14 Novembre 2018

Overview

1. Achromatic numbers
   - Achromatic number of a graph (Harary and Hedetniemi (1970))
   - Achromatic number of a 2-edge-colored graph
   - Achromatic number of a signed graph
   - Other achromatic numbers
2. NP-completeness

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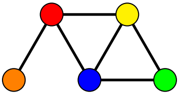
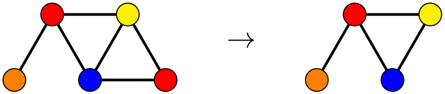
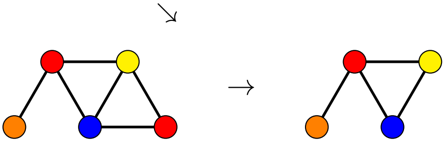




A surjective homomorphism is a sequence of identifications of non adjacent vertices.



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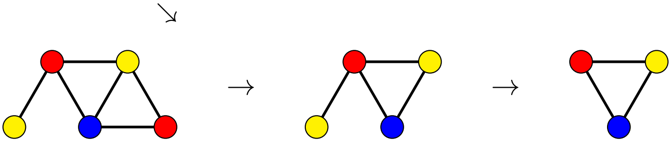

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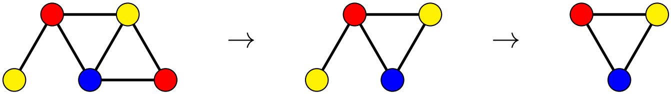

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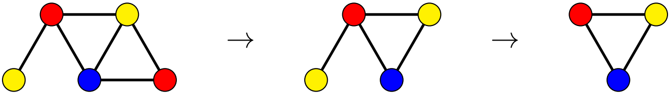

$\chi(G)$ is the order of the smallest clique we can reach from G by a homomorphism.

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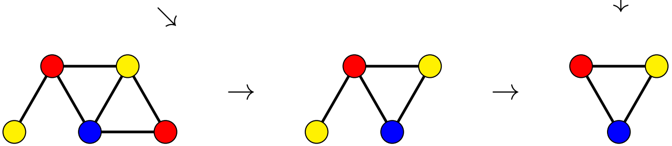

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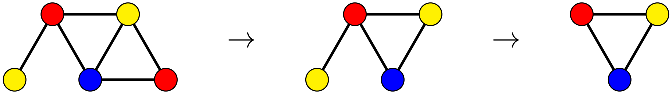

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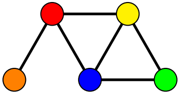

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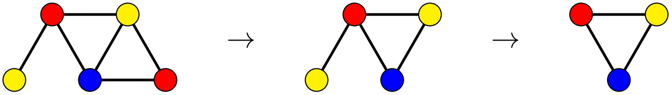

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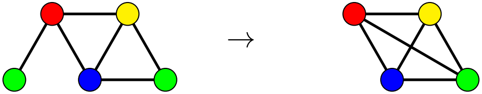

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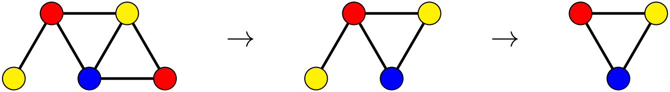

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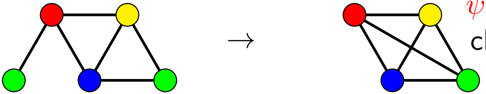

$\psi(G)$ is the order of the largest clique we can reach from G by a surjective homomorphism.

A surjective homomorphism is a sequence of identifications of non adjacent vertices. $\searrow$

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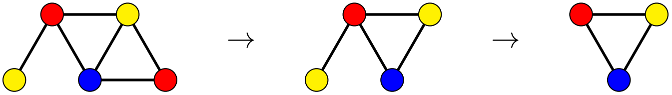

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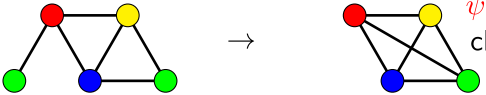

$\psi(G)$ is the order of the largest clique we can reach from G by a surjective homomorphism.

$\psi(G) = 4$

Consider the following algorithm:

Require: A graph G.
Ensure: Returns an integer $R(G)$.
while there exist two non adjacent vertices **do**
Choose randomly u and v such that $uv \notin E(G)$.
Identify u and v.
end while
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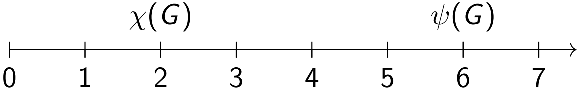

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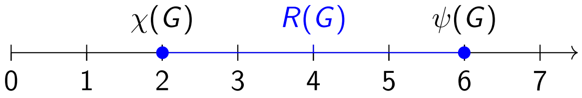

Problem: ACHROMATIC NUMBER
Instance: A graph G and an integer k
Question: Is $\psi(G) \geq k$?

Theorem (Yannakakis and Gavril, 1980)

The problem ACHROMATIC NUMBER *is NP-complete even for complements of bipartite graphs.*

Theorem (Bodlaender, 1989)

The problem ACHROMATIC NUMBER *is NP-complete even for graphs that are both connected interval graphs and co-graphs.*

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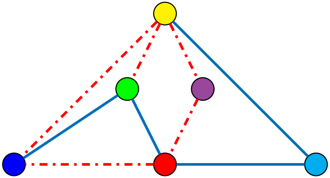
2. NP-completeness

Definition

A 2-edge-colored graph (G, C) is a simple undirected graph where each edge can be either positive or negative. C is the set of negative edges.



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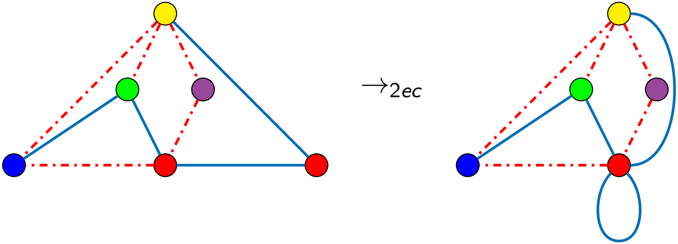

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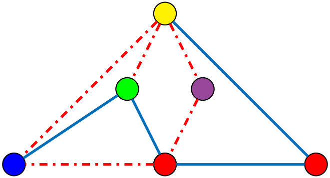
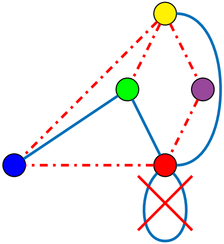


$\rightarrow_{2ec}$



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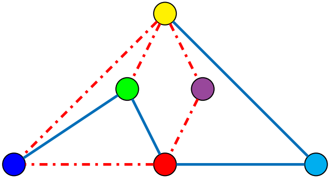

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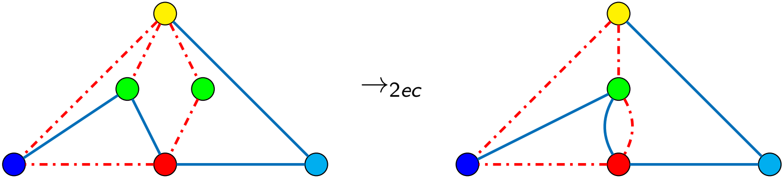

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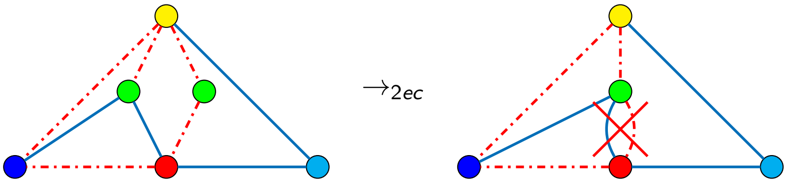

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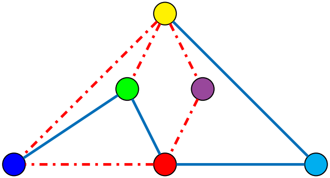

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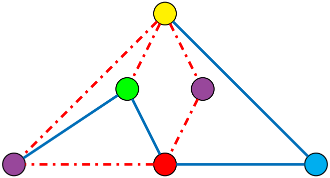

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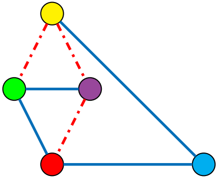

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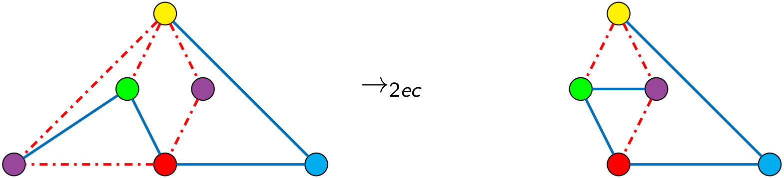

A 2-edge-colored clique.

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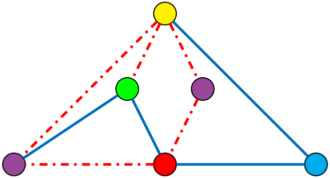

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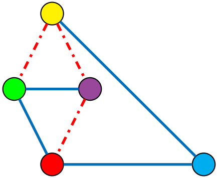

A 2-edge-colored clique.

$(G, C) \rightarrow_{2ec} (H, D) \iff$ there exists a surjective homomorphism from (G, C) to (H, D).

Definition

For a 2-edge-colored graph (G, C), we define and note:

- $\chi_2(G, C)$, the chromatic number of (G, C), is the order of the smallest 2-edge-colored clique (K, D) such that $(G, C) \to_{2ec} (K, D)$,
- $\psi_2(G, C)$, the achromatic number of (G, C), is the order of the largest 2-edge-colored clique (K, D) such that $(G, C) \to_{2ec} (K, D)$.

Problem: 2-EDGE-COLORED GRAPH ACHROMATIC NUMBER [2EC-AN]
Instance: A 2-edge-colored graph (G, C) and an integer k
Question: Is $\psi_2(G, C) \geq k$?

Theorem

The problem 2EC-AN is NP-complete even for graphs that are both connected interval graphs and co-graphs and for graphs that are complements of bipartite graphs.

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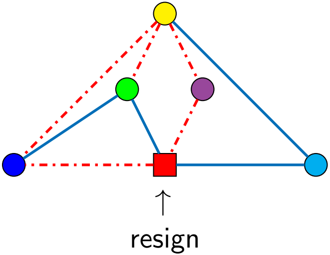
2. NP-completeness

Definition

A signed graph $[G, \Sigma]$ is a graph where each edge can be either positive or negative. Moreover we can resign at each vertex v. Resigning at v consists in inverting the signs of all edges incident with v. Σ is the set of negative edges.



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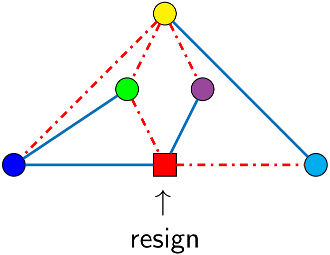

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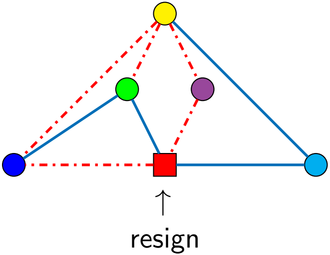

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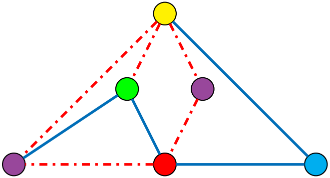
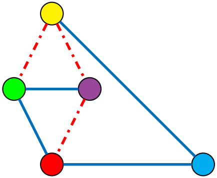


$\rightarrow_s$



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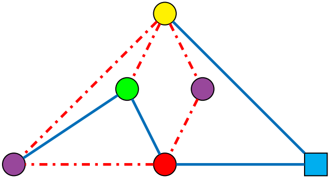

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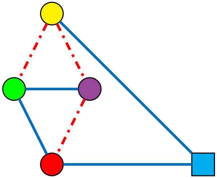

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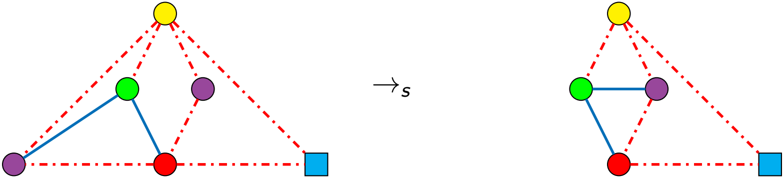

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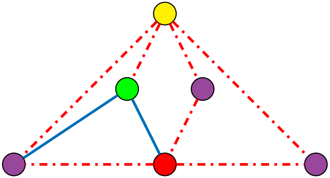

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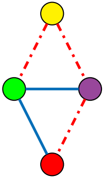

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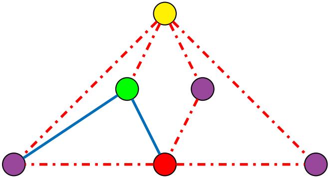

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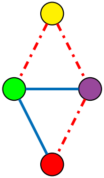

A signed clique.

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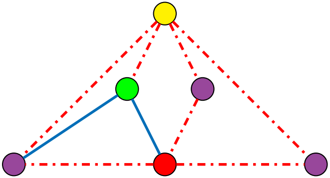

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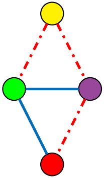
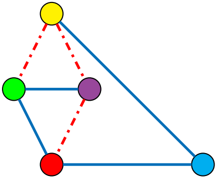
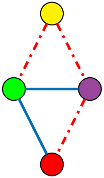


A signed clique.

$[G, \Sigma] \rightarrow_s [H, \Pi] \iff$ there exists a surjective homomorphism (identifications and resignings) from $[G, \Sigma]$ to $[H, \Pi]$.



This graph is a 2-edge-colored clique but not a signed clique.



This graph is a 2-edge-colored clique and a signed clique.

Definition

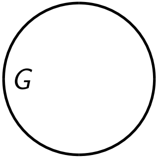
For a signed graph $[G, \Sigma]$, we define and note:

- $\chi_s[G, \Sigma]$, the chromatic number of $[G, \Sigma]$, is the order of the smallest signed clique $[K, \Pi]$ such that $[G, \Sigma] \to_s [K, \Pi]$,
- $\psi_s[G, \Sigma]$, the achromatic number of $[G, \Sigma]$, is the order of the largest signed clique $[K, \Pi]$ such that $[G, \Sigma] \to_s [K, \Pi]$.

Problem: SIGNED GRAPH ACHROMATIC NUMBER [SIGNED-AN]
Instance: A signed graph $[G, \Sigma]$ and an integer k
Question: Is $\psi_s[G, \Sigma] \geq k$?

Theorem

The problem SIGNED-AN *is NP-complete even for graphs that are both connected interval graphs and co-graphs and for graphs that are complements of bipartite graphs.*



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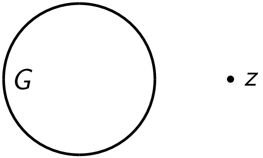

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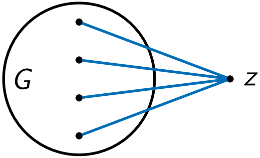

Problem: SIGNED GRAPH ACHROMATIC NUMBER [SIGNED-AN]
Instance: A signed graph $[G, \Sigma]$ and an integer k
Question: Is $\psi_s[G, \Sigma] \geq k$?

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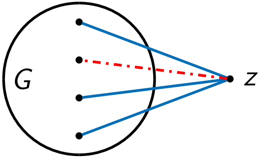

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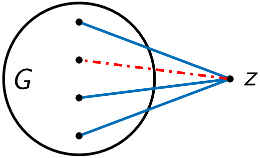

$$G \to K \iff [G + z, \varnothing] \to_s [K + z, \varnothing]$$

Overview

1. Achromatic numbers
   - Achromatic number of a graph (Harary and Hedetniemi (1970))
   - Achromatic number of a 2-edge-colored graph
   - Achromatic number of a signed graph
   - Other achromatic numbers
2. NP-completeness

Definition

For a graph G and a signed graph $[G, \Sigma]$:

- $\psi_{\text{min}}[G, \Sigma] = \min \{\psi_2(G, C) \mid (G, C) \in [G, \Sigma]\}$,
- $\psi_{\text{max}}[G, \Sigma] = \max \{\psi_2(G, C) \mid (G, C) \in [G, \Sigma]\}$,
- $\psi^{\text{2ec}}_{\text{max}}(G)$ is the order of the greatest 2ec clique (H, D) such that $(G, C) \to_{2ec} (H, D)$,
- $\psi^{\text{signed}}_{\text{max}}(G)$ is the order of the greatest signed clique $[H, \Pi]$ such that $[G, \Sigma] \to_s [H, \Pi]$.

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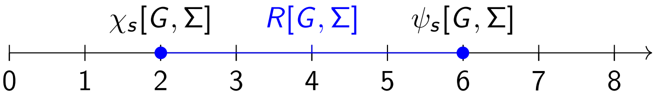

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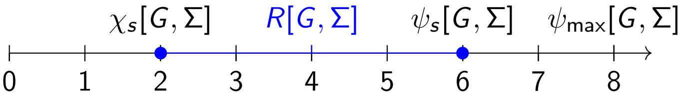


$\boxed{A \longleftarrow B \text{ means } A \leq B.}$

$$\begin{tikzcd}
 & \chi_2(G,C) \arrow[dl] & & \\
\chi_s[G,\Sigma] & \psi_{\mathrm{min}}[G,\Sigma] \arrow[l] & \psi_2(G,C) \arrow[l] \arrow[ul] & \psi_{\mathrm{max}}[G,\Sigma] \arrow[l] \arrow[dll] \\
 & \psi_s[G,\Sigma] \arrow[ul] & &
\end{tikzcd}$$

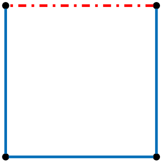
Figure: Relationship between some numbers for every signed graph $[G, \Sigma]$ and every 2-edge-colored graph $(G, C) \in [G, \Sigma]$.

Observation

In a 2-edge-colored clique (K, D), every two vertices u and v verify at least one of the following: either $uv \in E(K)$ or u and v are joined by a path $+-$ or $-+$.

Theorem (Naserasr, Rollová and Sopena, 2014)

In a signed clique $[K, \Pi]$, every two vertices u and v verify at least one of the following: either $uv \in E(K)$ or u and v belong to an UC_4.



UC_4

Corollary

A 2-edge-colored clique (K, D) has diameter 2.

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   - Other achromatic numbers
2. NP-completeness

Problem: SIGNED GRAPH MAX-ACHROMATIC NUMBER [SIGNED-MAX-AN]
Instance: A signed graph $[G, \Sigma]$ and an integer k
Question: Is $\psi_{\text{max}}[G, \Sigma] \geq k$?

Problem: GRAPH 2-EDGE-COLORED MAX-ACHROMATIC NUMBER [MAX-2EC-AN]
Instance: A graph G and an integer k
Question: Is $\psi_{\text{max}}^{\text{2ec}}(G) \geq k$?

Problem: GRAPH SIGNED MAX-ACHROMATIC NUMBER[MAX-SIGNED-AN]
Instance: A graph G and an integer k
Question: Is $\psi_{\text{max}}^{\text{signed}}(G) \geq k$?

Theorem

The following problems are NP-complete:

- SIGNED-MAX-AN, *even for connected diamond-free perfect graphs,*
- MAX-2EC-AN, *even for connected diamond-free perfect graphs,*
- MAX-SIGNED-AN, *even for connected perfect graphs.*

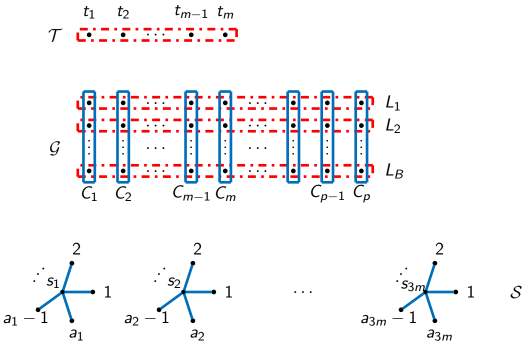
Problem: 3-PARTITION

Instance: A set $\mathcal{A} = \{a_1, \ldots, a_{3m}\} \in \mathbb{N}^{3m}$ and an integer B such that $\frac{B}{4} < a_i < \frac{B}{2}$ for every i, $1 \leq i \leq m$.

Question: Is there a partition $\{P_1, \ldots P_m\}$ of $\mathcal{A}$ such that $|P_i| = 3$ and $\sum_{a_j \in P_i} a_j = B$ for every i, $1 \leq i \leq m$?

Theorem (Garey and Johnson, 1990)

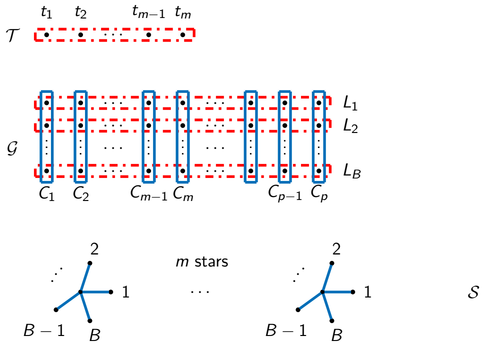
The problem 3-PARTITION is NP-complete.



 a positive complete subgraph on the vertices

 a negative complete subgraph on the vertices

 a complete bipartite positive graph between the left nodes and the right nodes



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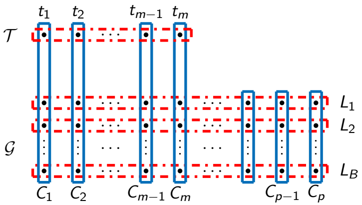

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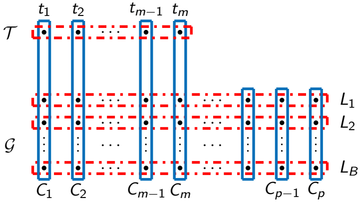

If 3-PARTITION has a solution then $\psi_{\max}[G,\Sigma] \geq |\mathcal{T}| + |\mathcal{G}|$.

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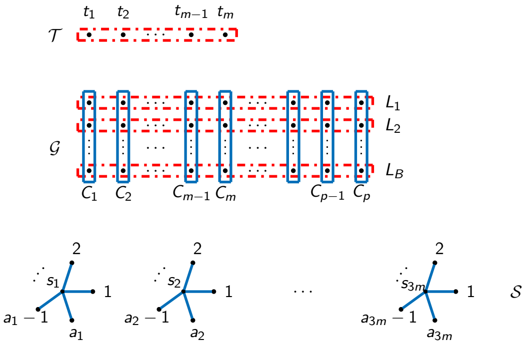

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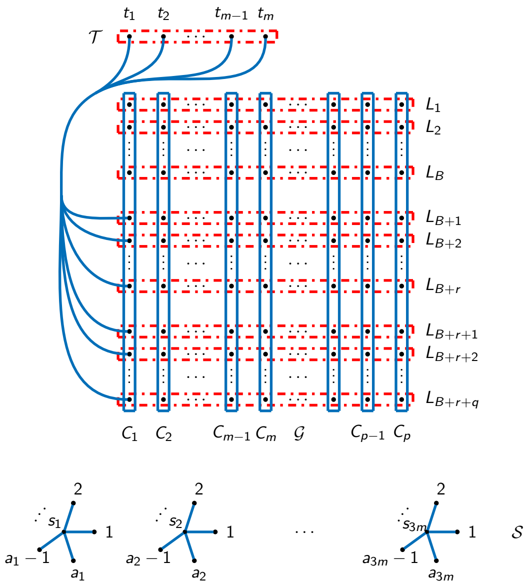
Claim

If $(G, C) \rightarrow_{2ec} (K, D)$ where (K, D) is a 2-edge-colored clique of size greater than $|\mathcal{T}| + |\mathcal{G}|$ then $G \rightarrow K'$ where K' has order greater than $|\mathcal{T}| + |\mathcal{G}|$ and K' has diameter 2.


t_1 t_2 t_{m-1} t_m
$\mathcal{T}$
L_1
L_2
L_B
L_{B+1}
L_{B+2}
L_{B+r}
L_{B+r+1}
L_{B+r+2}
L_{B+r+q}
C_1 C_2 C_{m-1} C_m $\mathcal{G}$ C_{p-1} C_p
2
s_1
1
$a_1 - 1$
a_1
2
s_2
1
$a_2 - 1$
a_2
$\cdots$
2
s_{3m}
1
$a_{3m} - 1$
a_{3m}
$\mathcal{S}$

	Ordinary graphs	2-edge-colored graphs	Signed graphs
ψ	NP-complete		
ψ_2		NP-complete	
ψ_s			NP-complete
ψ_{max}	NP-complete		NP-complete
ψ_{min}	Π_2 (complete ?)		Π_2 (complete ?)
$\psi_{\text{max}}^{\text{2ec}}$	NP-complete		
$\psi_{\text{max}}^{\text{signed}}$	NP-complete		

Table: Decision problems related to achromatic numbers.

	Ordinary graphs	2-edge-colored graphs	Signed graphs
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ψ_s			NP-complete
ψ_{max}	NP-complete		NP-complete
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$\psi_{\text{max}}^{\text{2ec}}$	NP-complete		
$\psi_{\text{max}}^{\text{signed}}$	NP-complete		

Table: Decision problems related to achromatic numbers.

Thank you for your attention!

References I

Hans L. Bodlaender.

Achromatic number is NP-complete for cographs and interval graphs.

Information Processing Letters, 31(3):135–138, 1989.

Michael R. Garey and David S. Johnson.

Computers and Intractability; A Guide to the Theory of NP-Completeness.

W. H. Freeman & Co., New York, NY, USA, 1990.

Frank Harary and Stephen Hedetniemi.

The achromatic number of a graph.

Journal of Combinatorial Theory, 8(2):154–161, 1970.

References II

Reza Naserasr, Edita Rollová, and Éric Sopena.
Homomorphisms of signed graphs.
Journal of Graph Theory, 79(3):178–212, 2015.

M. Yannakakis and F. Gavril.
Edge dominating sets in graphs.
SIAM Journal on Applied Mathematics, 38(3):364–372, 1980.