Achromatic number of signed graphs

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Overview

1. Achromatic numbers
   - Achromatic number of a graph (Harary and Hedetniemi (1970))
   - Achromatic number of a 2-edge-colored graph
   - Achromatic number of a signed graph
   - Other achromatic numbers

2. NP-completeness
Overview

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2. NP-completeness
Achromatic numbers

A surjective homomorphism is a sequence of identifications of non adjacent vertices.

A clique is a graph in which we cannot identify vertices.

χ(G) is the order of the smallest clique we can reach from G by a homomorphism.

χ(G) = 3

ψ(G) is the order of the largest clique we can reach from G by a surjective homomorphism.

ψ(G) = 4
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A chromatic number is a sequence of identifications of non-adjacent vertices. A clique is a graph in which we cannot identify vertices.

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Consider the following algorithm:

**Require:** A graph $G$.

**Ensure:** Returns an integer $R(G)$.

while there exist two non adjacent vertices do

Choose randomly $u$ and $v$ such that $uv \notin E(G)$.

Identify $u$ and $v$.

end while

return $|G|$
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Problem: Achromatic number
Instance: A graph $G$ and an integer $k$
Question: Is $\psi(G) \geq k$?

Theorem (Yannakakis and Gavril, 1980)

The problem Achromatic number is NP-complete even for complements of bipartite graphs.

Theorem (Bodlaender, 1989)

The problem Achromatic number is NP-complete even for graphs that are both connected interval graphs and co-graphs.
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2. NP-completeness
A 2-edge-colored graph \((G, C)\) is a simple undirected graph where each edge can be either positive or negative. \(C\) is the set of negative edges.
Achromatic numbers

NP-completeness

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\rightarrow_{2ec}\quad (G, C) \rightarrow_{2ec} (H, D) \iff \text{there exists a surjective homomorphism from } (G, C) \text{ to } (H, D).
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Definition

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A 2-edge-colored clique.

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\[(G, C) \xrightarrow{2\text{ec}} (H, D) \iff \text{there exists a surjective homomorphism from } (G, C) \text{ to } (H, D).\]
Achromatic numbers

NP-completeness

Definition

For a 2-edge-colored graph \((G, C)\), we define and note:

- \(\chi_2(G, C)\), the chromatic number of \((G, C)\), is the order of the smallest 2-edge-colored clique \((K, D)\) such that \((G, C) \rightarrow_{2ec} (K, D)\).

- \(\psi_2(G, C)\), the achromatic number of \((G, C)\), is the order of the largest 2-edge-colored clique \((K, D)\) such that \((G, C) \rightarrow_{2ec} (K, D)\).
Problem: 2-EDGE-COLORED GRAPH ACHROMATIC NUMBER

[2EC-AN]

Instance: A 2-edge-colored graph \((G, C)\) and an integer \(k\)

Question: Is \(\psi_2(G, C) \geq k?\)

Theorem

The problem 2EC-AN is NP-complete even for graphs that are both connected interval graphs and co-graphs and for graphs that are complements of bipartite graphs.
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2 NP-completeness
A signed graph \([G, \Sigma]\) is a graph where each edge can be either positive or negative. Moreover, we can resign at each vertex \(v\). Resigning at \(v\) consists in inverting the signs of all edges incident with \(v\). \(\Sigma\) is the set of negative edges.
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A signed clique.

\[\rightarrow_s\]
Definition

A signed graph $[G, \Sigma]$ is a graph where each edge can be either positive or negative. Moreover, we can resign at each vertex $v$. Resigning at $v$ consists in inverting the signs of all edges incident with $v$. $\Sigma$ is the set of negative edges.

A signed clique.

$[G, \Sigma] \rightarrow_s [H, \Pi] \iff$ there exists a surjective homomorphism (identifications and resignings) from $[G, \Sigma]$ to $[H, \Pi]$. 
This graph is a 2-edge-colored clique but not a signed clique.

This graph is a 2-edge-colored clique and a signed clique.
Definition

For a signed graph $[G, \Sigma]$, we define and note:

- $\chi_s[G, \Sigma]$, the **chromatic number** of $[G, \Sigma]$, is the order of the **smallest** signed clique $[K, \Pi]$ such that $[G, \Sigma] \rightarrow_s [K, \Pi]$,

- $\psi_s[G, \Sigma]$, the **achromatic number** of $[G, \Sigma]$, is the order of the **largest** signed clique $[K, \Pi]$ such that $[G, \Sigma] \rightarrow_s [K, \Pi]$. 
Problem: **Signed graph achromatic number**

[Signed-an]

Instance: A signed graph \([G, \Sigma]\) and an integer \(k\)

Question: Is \(\psi_s[G, \Sigma] \geq k?\)

Theorem

The problem **Signed-an** is NP-complete even for graphs that are both connected interval graphs and co-graphs and for graphs that are complements of bipartite graphs.
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2. NP-completeness
Definition

For a graph $G$ and a signed graph $[G, \Sigma]$:

- $\psi_{\min}[G, \Sigma] = \min \{ \psi_2(G, C) \mid (G, C) \in [G, \Sigma] \}$,
- $\psi_{\max}[G, \Sigma] = \max \{ \psi_2(G, C) \mid (G, C) \in [G, \Sigma] \}$,
- $\psi_{2ec}^{\max}(G)$ is the order of the greatest 2ec clique $(H, D)$ such that $(G, C) \rightarrow_{2ec} (H, D)$,
- $\psi_{\max}^{\text{signed}}(G)$ is the order of the greatest signed clique $[H, \Pi]$ such that $[G, \Sigma] \rightarrow_s [H, \Pi]$. 
Achromatic numbers

NP-completeness

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- $\psi^{\text{signed}}(G)$ is the order of the greatest signed clique $[H, \Pi]$ such that $[G, \Sigma] \rightarrow_s [H, \Pi]$. 

\[ \chi_s[G, \Sigma] \quad R[G, \Sigma] \quad \psi_s[G, \Sigma] \]
Achromatic numbers

NP-completeness

Definition

For a graph $G$ and a signed graph $[G, \Sigma]$:

- $\psi_{\min}[G, \Sigma] = \min \{ \psi_2(G, C) \mid (G, C) \in [G, \Sigma] \}$,
- $\psi_{\max}[G, \Sigma] = \max \{ \psi_2(G, C) \mid (G, C) \in [G, \Sigma] \}$,
- $\psi_{2\text{ec}}(G)$ is the order of the greatest $2\text{ec}$ clique $(H, D)$ such that $(G, C) \rightarrow_{2\text{ec}} (H, D)$,
- $\psi_{\text{signed}}(G)$ is the order of the greatest signed clique $[H, \Pi]$ such that $[G, \Sigma] \rightarrow_s [H, \Pi]$.

\[
\begin{align*}
\chi_s[G, \Sigma] & \quad R[G, \Sigma] & \quad \psi_s[G, \Sigma] & \quad \psi_{\max}[G, \Sigma] \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{align*}
\]
Achromatic numbers

NP-completeness

\[ \chi_s[G, \Sigma] \leq \psi_{\min}[G, \Sigma] \leq \psi_2(G, C) \leq \psi_{\max}[G, \Sigma] \]

Figure: Relationship between some numbers for every signed graph \([G, \Sigma]\) and every 2-edge-colored graph \((G, C) \in [G, \Sigma]\).
Observation

In a 2-edge-colored clique \((K, D)\), every two vertices \(u\) and \(v\) verify at least one of the following: either \(uv \in E(K)\) or \(u\) and \(v\) are joined by a path \(+-\) or \(-+\).

Theorem (Naserasr, Rollová and Sopena, 2014)

In a signed clique \([K, \Pi]\), every two vertices \(u\) and \(v\) verify at least one of the following: either \(uv \in E(K)\) or \(u\) and \(v\) belong to an \(UC_4\).
Corollary

A 2-edge-colored clique \((K, D)\) has diameter 2.
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2 NP-completeness
Problem: **Signed graph max-achromatic number**

\[
\text{Instance: A signed graph } [G, \Sigma] \text{ and an integer } k
\]

\[
\text{Question: Is } \psi_{\text{max}}[G, \Sigma] \geq k? \]

Problem: **Graph 2-edge-colored max-achromatic number**

\[
\text{Instance: A graph } G \text{ and an integer } k
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Problem: **Graph signed max-achromatic number**

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\text{Instance: A graph } G \text{ and an integer } k
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\text{Question: Is } \psi_{\text{max}}^{\text{signed}}(G) \geq k? \]

**Theorem**

The following problems are NP-complete:

- **Signed-max-an**, even for connected diamond-free perfect graphs,
- **Max-2ec-an**, even for connected diamond-free perfect graphs,
- **Max-signed-an**, even for connected perfect graphs.
Problem: 3-PARTITION

Instance: A set $A = \{a_1, \ldots, a_{3m}\} \in \mathbb{N}^{3m}$ and an integer $B$ such that $\frac{B}{4} < a_i < \frac{B}{2}$ for every $i, 1 \leq i \leq m$.

Question: Is there a partition $\{P_1, \ldots, P_m\}$ of $A$ such that $|P_i| = 3$ and $\sum_{a_j \in P_i} a_j = B$ for every $i, 1 \leq i \leq m$?

Theorem (Garey and Johnson, 1990)

The problem 3-PARTITION is NP-complete.
Achromatic numbers

NP-completeness

If 3-partition has a solution then $\psi_{\text{max}}[G, \Sigma] \geq |T| + |G|$.

- A positive complete subgraph on the vertices
- A negative complete subgraph on the vertices
- A complete bipartite positive graph between the left nodes and the right nodes
Achromatic numbers

NP-completeness

If \(3\)-partition has a solution then
\[
\psi_{\text{max}}[G, \Sigma] \geq |T| + |G|.
\]

\[
T = [t_1, t_2, t_{m-1}, t_m]
\]

\[
G = [C_1, C_2, C_{m-1}, C_m, C_{p-1}, C_p, L_1, L_2, L_B]
\]

\[
S = \text{a positive complete subgraph on the vertices}
\]

\[
\text{a negative complete subgraph on the vertices}
\]

\[
\text{a complete bipartite positive graph between the left nodes and the right nodes}
\]

\[
s_1 s_2 s_3 m 1 2 a_1 - 1 a_1 1 2 a_2 - 1 a_2 1 2 a_3 - 1 a_3 m - 1 a_3 m 2 B - 1 B 1 \]

A positive complete subgraph on the vertices

A negative complete subgraph on the vertices

A complete bipartite positive graph between the left nodes and the right nodes
If \(3\text{-PARTITION}\) has a solution then \(\psi_{\text{max}}[G, \Sigma] \geq |\mathcal{T}| + |\mathcal{G}|.\)

- A positive complete subgraph on the vertices
- A negative complete subgraph on the vertices
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Achromatic numbers

NP-completeness

If 3-partition has a solution then
\[ \psi_{\text{max}}[G, \Sigma] \geq |T| + |G|. \]

- A positive complete subgraph on the vertices
- A negative complete subgraph on the vertices
- A complete bipartite positive graph between the left nodes and the right nodes
Claim

If \((G, C) \rightarrow_{2\text{ec}} (K, D)\) where \((K, D)\) is a 2-edge-colored clique of size greater than \(|\mathcal{T}| + |\mathcal{G}|\) then \(G \rightarrow K'\) where \(K'\) has order greater than \(|\mathcal{T}| + |\mathcal{G}|\) and \(K'\) has diameter 2.
Achromatic numbers
NP-completeness
Ordinary graphs
2-edge-colored graphs
Signed graphs

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Table: Decision problems related to achromatic numbers.
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**Table:** Decision problems related to achromatic numbers.

Thank you for your attention!
Hans L. Bodlaender.
Achromatic number is NP-complete for cographs and interval graphs.

Michael R. Garey and David S. Johnson.
*Computers and Intractability; A Guide to the Theory of NP-Completeness*.

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Reza Naserasr, Edita Rollová, and Éric Sopena.
Homomorphisms of signed graphs.

M. Yannakakis and F. Gavril.
Edge dominating sets in graphs.