Eternal Domination in Grids

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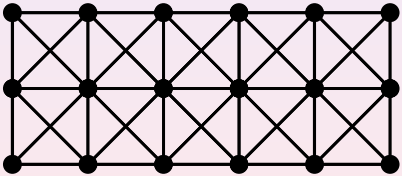
Eternal Domination Game [Burger et al, 2004 ; Goddard et al, 2005]

Overview

- k guards vs. 1 attacker
- **Each turn :** attacker attacks any vertex and guards may move to neighbours.
- Guards must move to occupy a **dominating** set containing last attacked vertex. If they can do so **eternally**, they win. Otherwise, they lose.
- $\gamma_{all}^{\infty}(G)$: min. # guards needed to guarantee they win in G.



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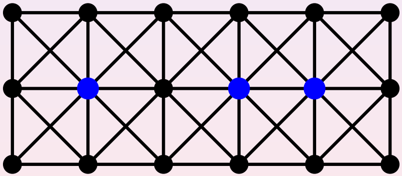

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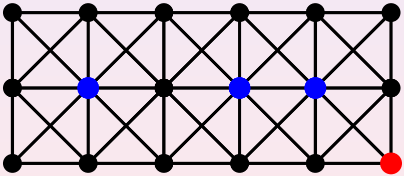

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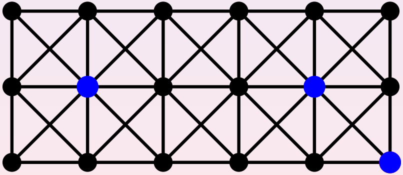

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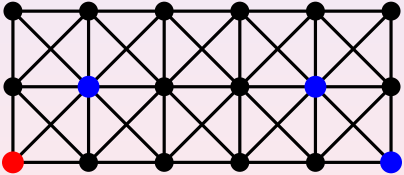

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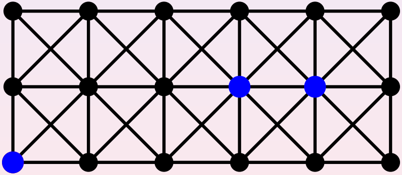

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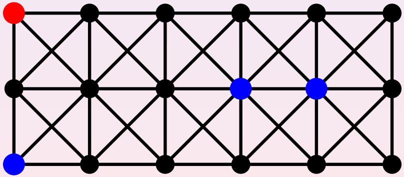

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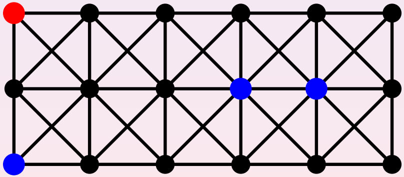

Attacker wins!

Eternal Domination Game [Burger et al, 2004 ; Goddard et al, 2005]

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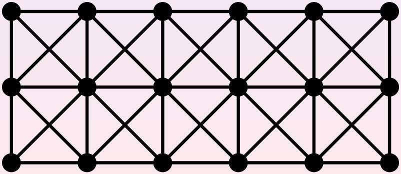

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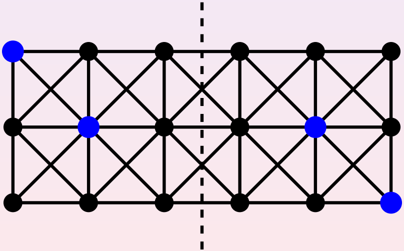

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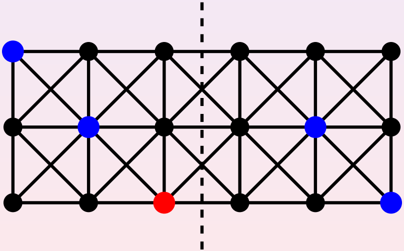

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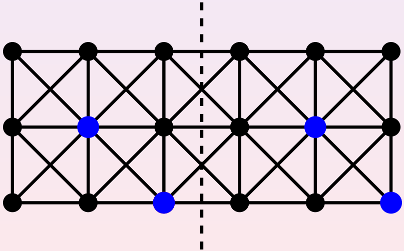

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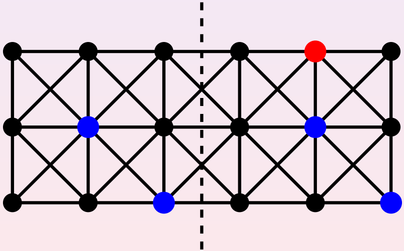

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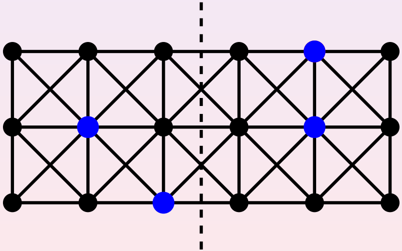

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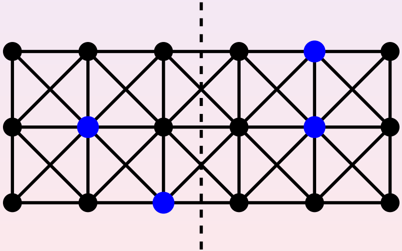

Guards win!

State of the art

- Deciding whether $\gamma_{all}^{\infty}(G) \leq k$ is NP-hard [Bard et al, 2017].
- For all G, $\gamma(G) \leq \gamma_{all}^{\infty}(G) \leq \alpha(G)$ [Goddard et al, 2005].
- Paths and cycles are easy ($\gamma_{all}^{\infty}(P_n) = \lceil \frac{n}{2} \rceil$, $\gamma_{all}^{\infty}(C_n) = \lceil \frac{n}{3} \rceil$) [Goddard et al, 2005].
- Linear-time algorithm for trees [Klostermeyer, MacGillivray, 2009].
- $\gamma_{all}^{\infty}(G) = \alpha(G)$ for all proper interval graphs G [Braga et al, 2015].
- Recently studied in digraphs [Bagan et al, 2018].

Cartesian Grids

- $\gamma_{all}^{\infty}(P_2 \Box P_n) = \lceil \frac{2n}{3} \rceil$ [Goldwasser et al, 2013].
- $\lceil \frac{4n}{5} \rceil + 1 \leq \gamma_{all}^{\infty}(P_3 \Box P_n) \leq \lceil \frac{4n}{5} \rceil + 5$ [Messinger, 2017].
- $\gamma_{all}^{\infty}(P_4 \Box P_n)$ is known [Beaton et al, 2014] and bounds for $\gamma_{all}^{\infty}(P_5 \Box P_n)$ exist [van Bommel et al, 2016].

Theorem [Lamprou et al, 2017]

$\gamma_{all}^{\infty}(P_n \Box P_m) = \gamma(P_n \Box P_m) + O(n + m)$.

Eternal Domination in Strong Grids [M., Nisse, Pérennes, 2018]

Note that $\gamma(P_n \boxtimes P_m) = \lceil \frac{mn}{9} \rceil$ and $\alpha(G) = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ and so

$$\left\lceil \frac{mn}{9} \right\rceil \leq \gamma_{all}^{\infty}(P_n \boxtimes P_m) \leq \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil .$$

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Theorem [M., Nisse, Pérennes, 2018]

For all $m \geq n$,

$\lfloor \frac{mn}{9} \rfloor + \Omega(n+m) = \gamma_{all}^{\infty}(P_n \boxtimes P_m) = \lceil \frac{mn}{9} \rceil + O(m\sqrt{n})$

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Theorem [M., Nisse, Pérennes, 2018]

For all $m \geq n$ such that $n \mod 3 = m \mod 3 = 0$,

$$\gamma(P_n \boxtimes P_m) + \Omega(n+m) = \gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + O(m\sqrt{n})$$

Eternal domination in the torus $C_n \boxtimes C_m$

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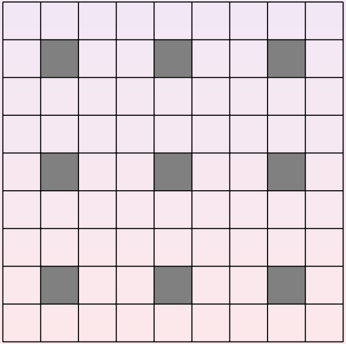

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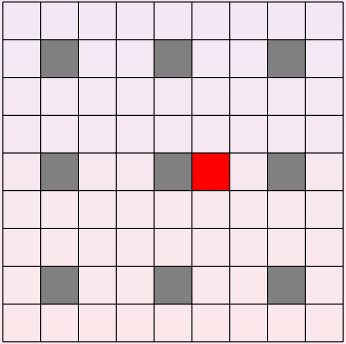

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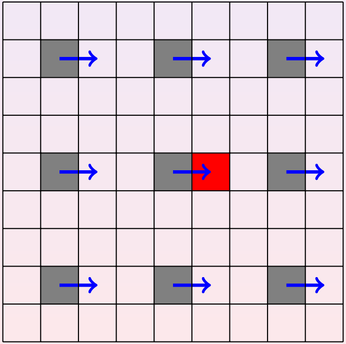

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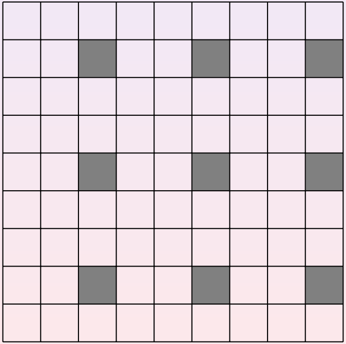

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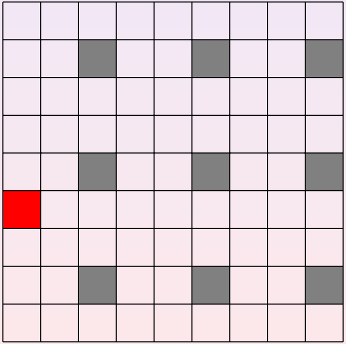

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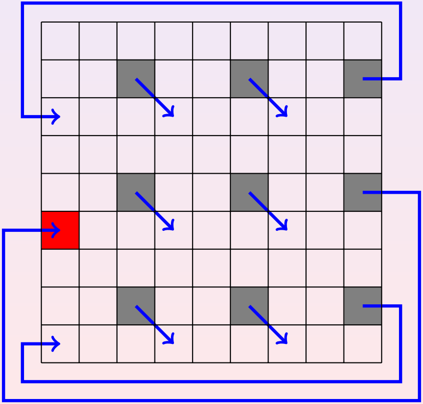

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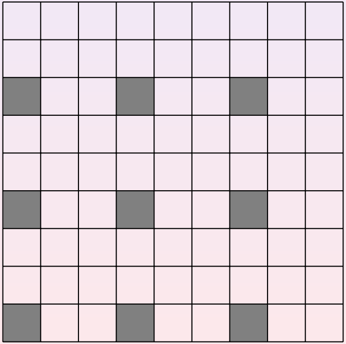

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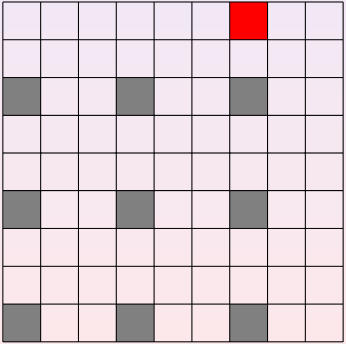

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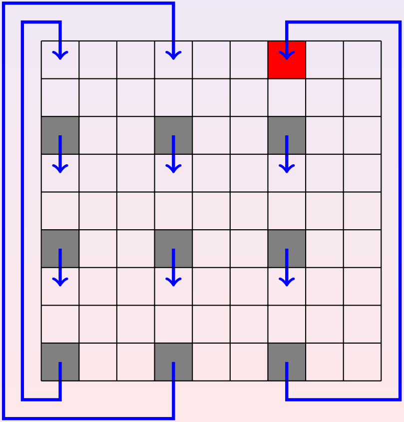

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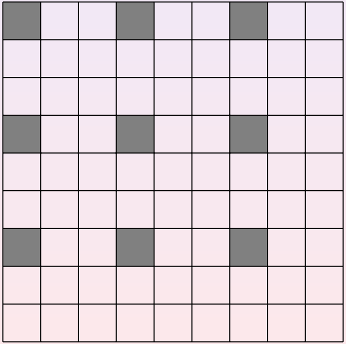
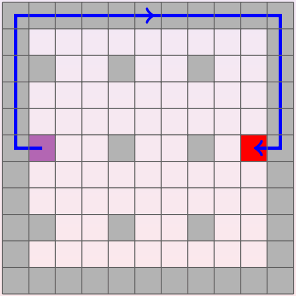


Easy in the torus because we can wrap around $\rightarrow$ impossible in the grid !

Back to the Grid : Key Lemma

Teleportation

If there are α guards on each border vertex, then $\beta \leq \alpha$ guards may teleport using the borders of the grid.



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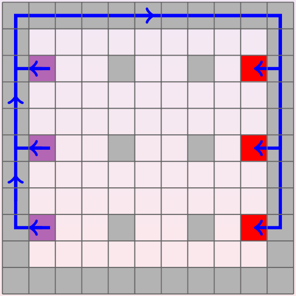


Upper Bound Idea of Proof ($\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \lceil \frac{mn}{9} \rceil + O(m\sqrt{n})$)

Configuration

Multi-set $C = \{v_i \mid 1 \leq i \leq k\}$ giving the positions of the k guards.

Configurations of the winning strategy : $SetWinConf$

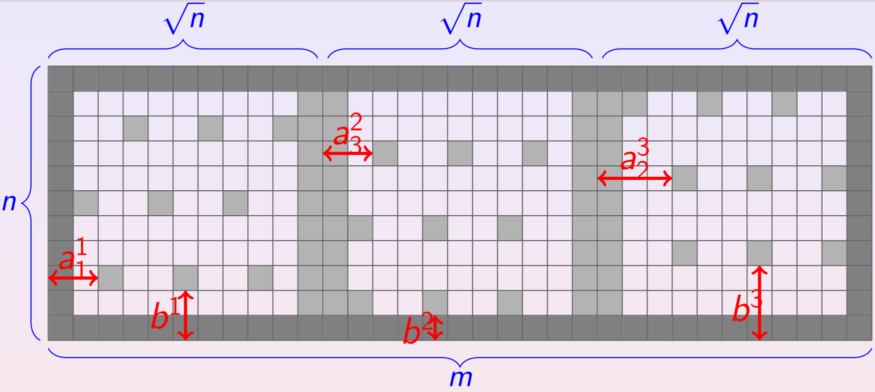
Set of configurations that dominate the grid.

Attacks split into 3 types : Horizontal, Vertical, and Diagonal.

We show that, for any attack, the guards can move from a configuration $C \in SetWinConf$ to a configuration $C' \in SetWinConf$.

C and C' are said to be compatible in this case.

Configuration $C \in SetWinConf$

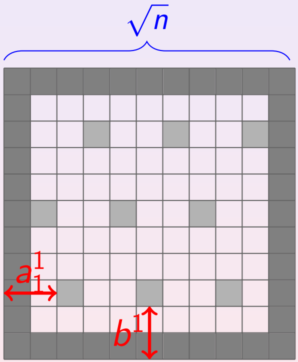


1 guard on vertices in light gray.

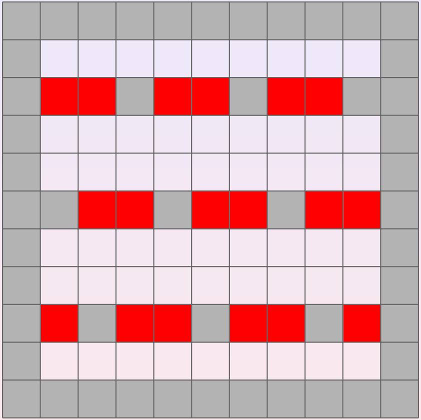
$O(\sqrt{n})$ guards on vertices in dark gray.

$\gamma(P_n \boxtimes P_m) + O(m\sqrt{n})$ guards total.

Block of Configuration $C \in SetWinConf$

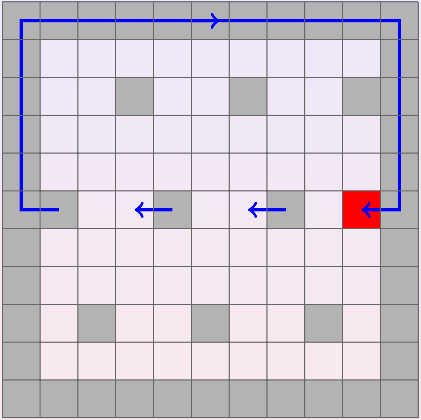


Horizontal Attacks



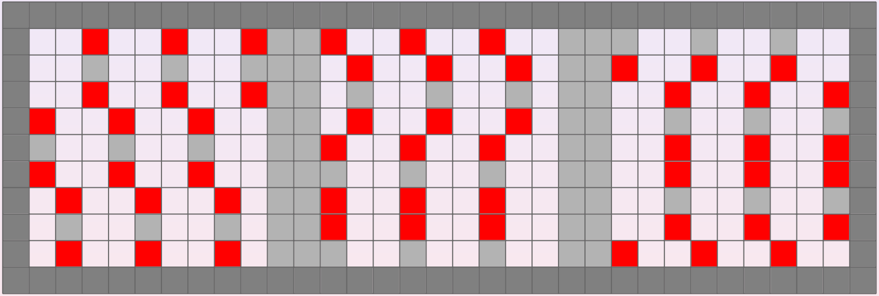
Horizontal attacks may only occur at vertices in red.

Horizontal Attacks - attack at red vertex



Only guards in same row and block move (except for borders maybe).

Vertical Attacks



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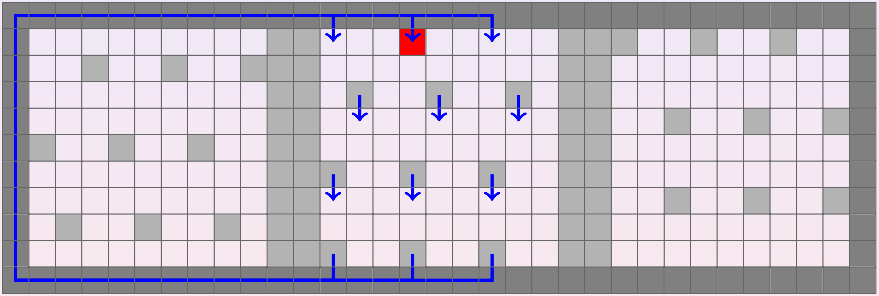

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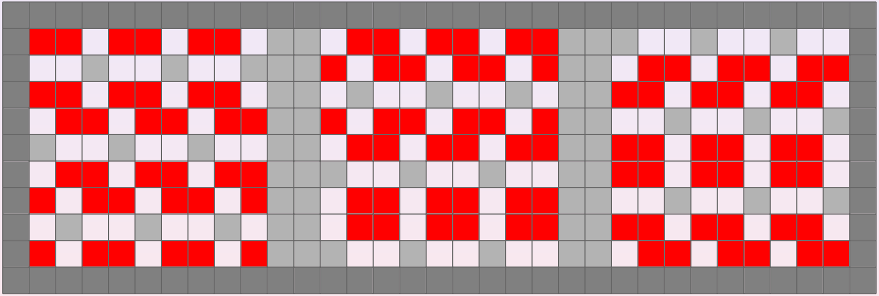
Diagonal attacks



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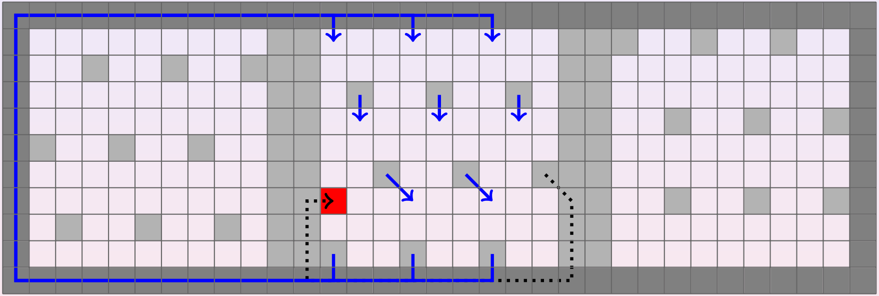
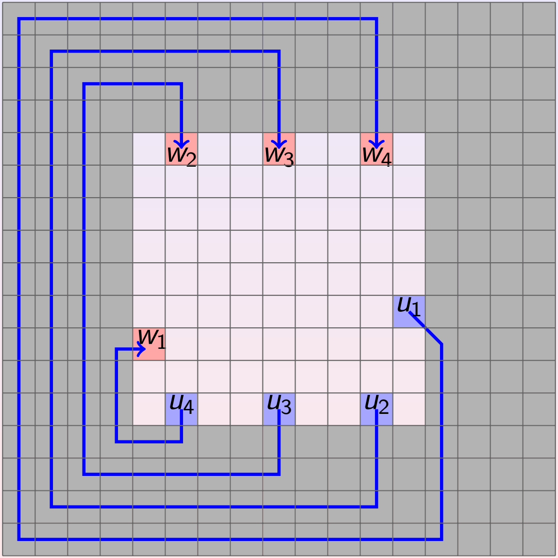


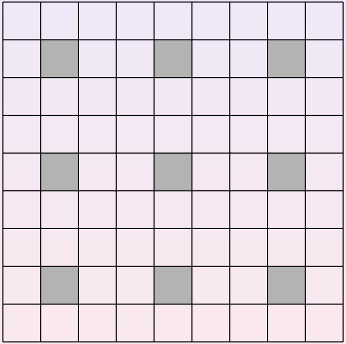
Guards in closest row (and block) move like in Horizontal and Vertical case at once and the rest in the same block move like in Vertical case.

At most 1 guard at each vertex



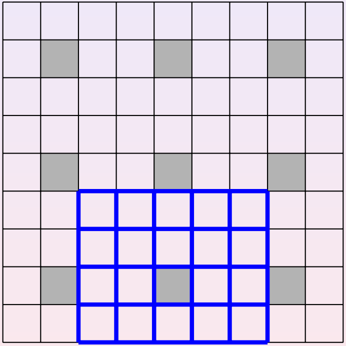
Lower Bound Idea of Proof $(\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + \Omega(m+n))$

At least 2 guards needed in each 4×5 subgrid on the border.



Lower Bound Idea of Proof $(\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + \Omega(m+n))$

At least 2 guards needed in each 4×5 subgrid on the border.



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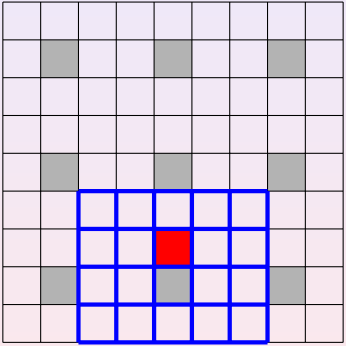

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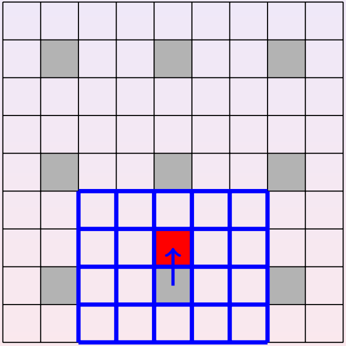

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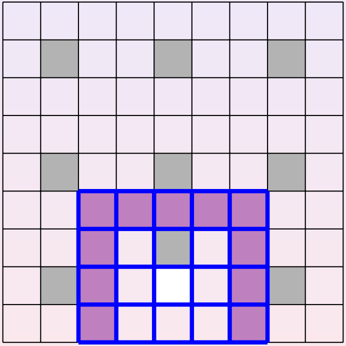

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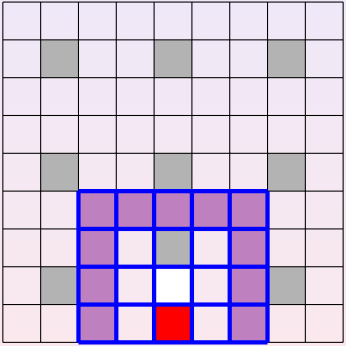

Counting argument leads to result.

Further Work

- Tighten bounds for strong grids.
- For all Cayley graphs G obtainable from abelian groups, $\gamma_{all}^{\infty}(G) = \gamma(G)$ [Goddard et al, 2005].
  - Generalize our technique to prove $\gamma_{all}^{\infty}(H) = \gamma(H) + o(\gamma(H))$ for truncated Cayley graphs H obtained from abelian groups.
- All-guards move model is *NP*-hard but it's not known if it's in *NP*.
  - Is it *PSPACE*-complete? *EXPTIME*-complete?

Thanks !