Explicit 3-colorings for exponential graphs

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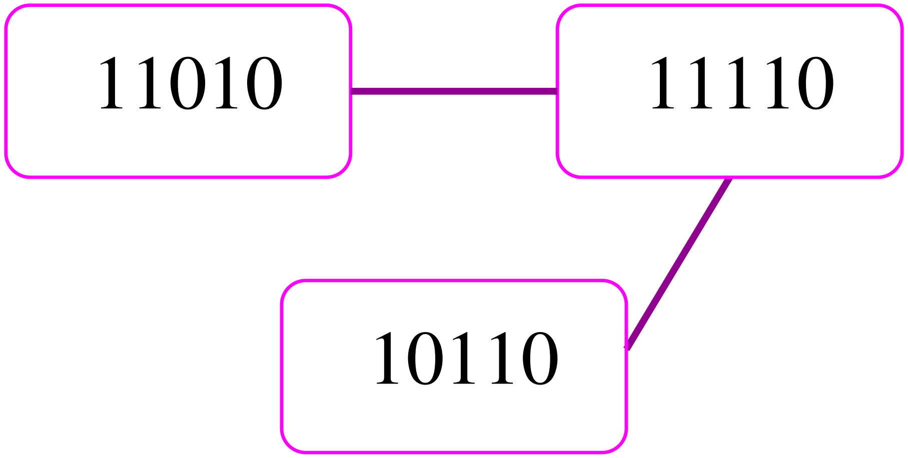
November 14, 2018

The Hypercube



Vertices: All binary strings of length n.

Edges: Edge between two vertices (strings) if they differ in exactly one position.

The Hypercube is Bipartite

How can we find a bipartition? Use Breadth-First-Search.

This takes $O(2^n)$, but is polynomial in size of hypercube.

Now, suppose we are given the vertices one-by-one by an adversary.

Then can we assign each vertex to a side of the bipartition in time poly(n)?

Bipartition of the Hypercube

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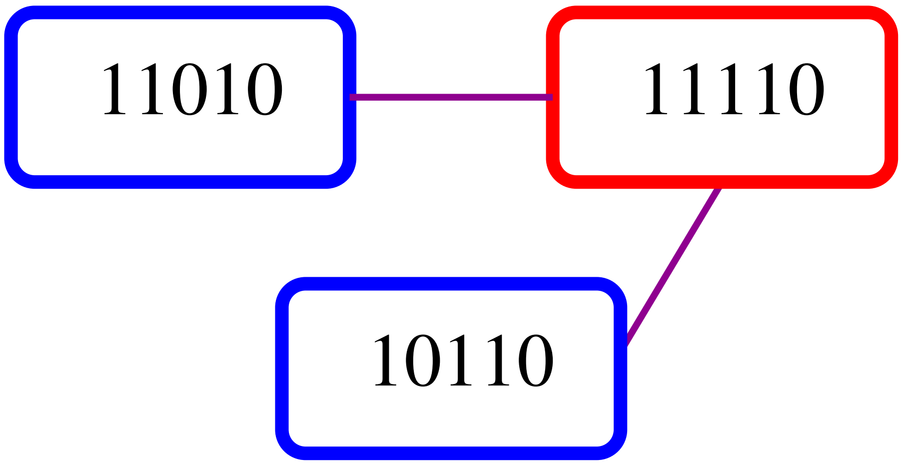
Now, suppose we are given the vertices one-by-one by an adversary.

Then can we assign each vertex to a side of the bipartition in time poly(n)?

All vertices with even number of 1's in EVEN, and
All vertices with odd number of 1's in ODD.

“Explicit” Bipartition of the Hypercube



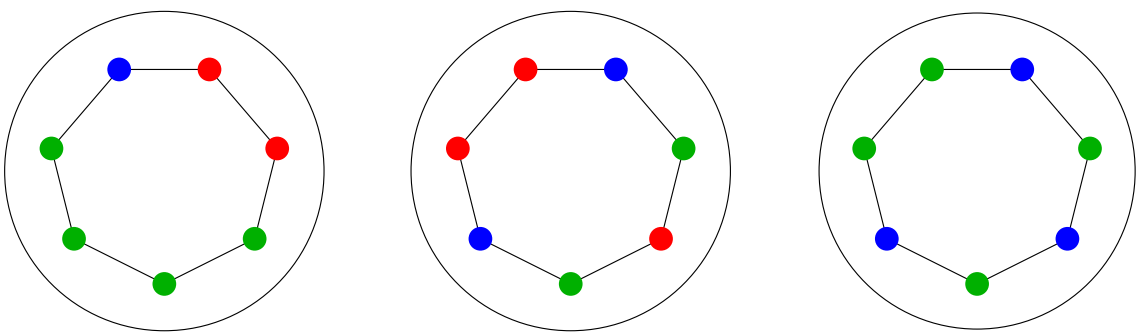
We can determine to which side a vertex belongs in time $O(n)$.

Explicit bipartition: reason why a vertex belongs to one side or the other.

Now we describe another bipartite graph ... First, a definition.

The 3-Coloring Exponential Graph K

- Let C_n denote odd cycle on n *nodes* (i.e., n is odd).
- Vertices of $K = K_3^{C_n}$ are all 3-colorings of C_n (i.e., 3^n vertices).
- Note that 3-colorings can be non-proper.



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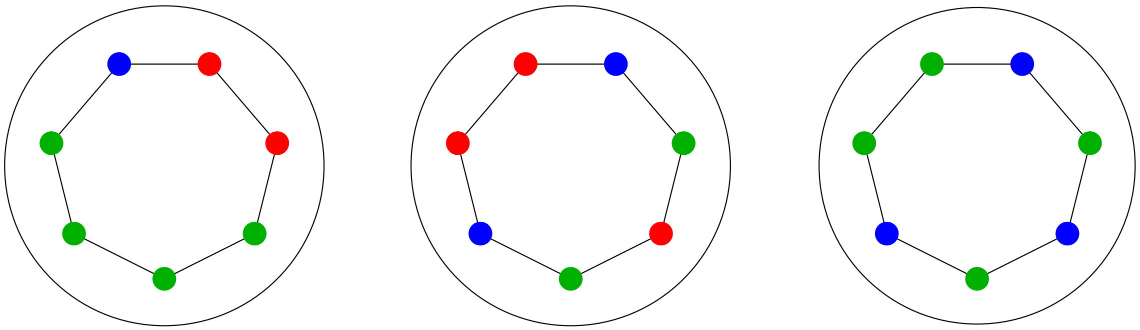
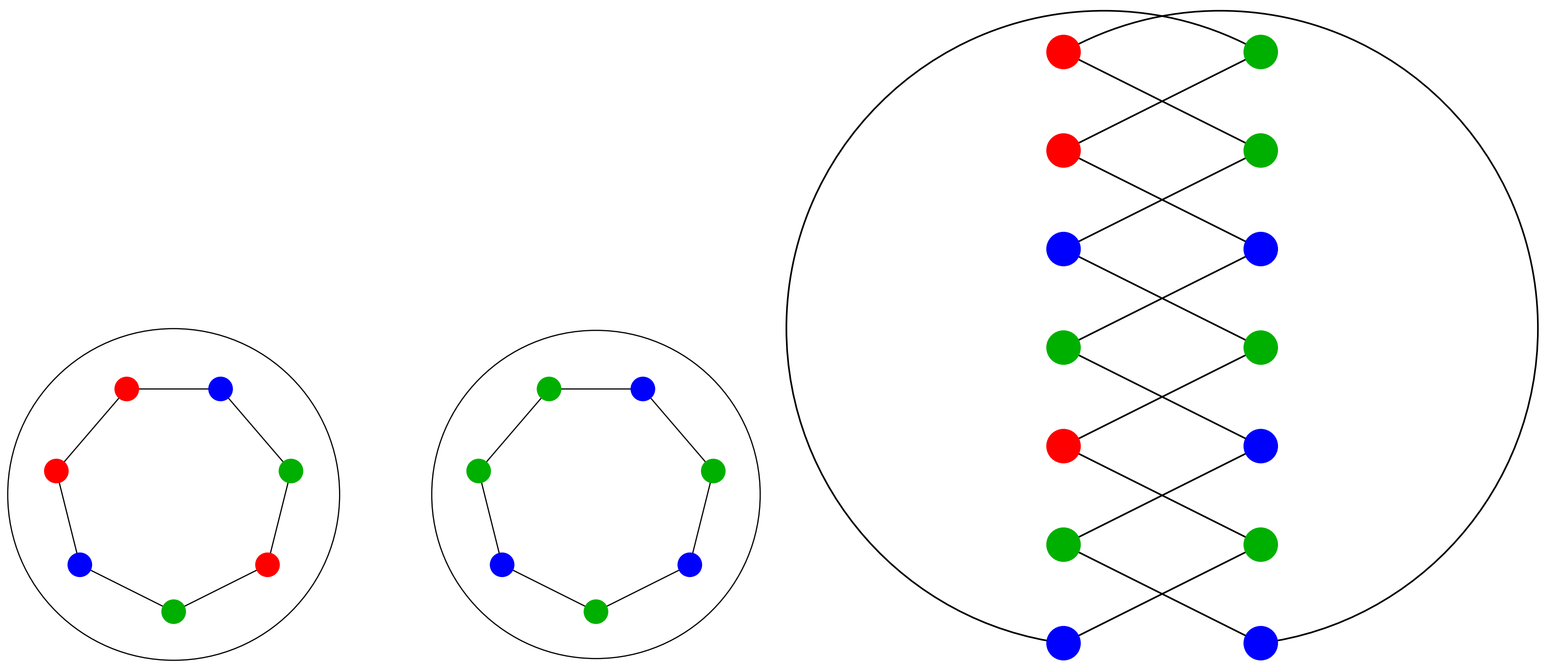


- Add an edge between two 3-colorings if the bipartite graph between them is properly colored.

$\forall ij \in E(C_n)$, check if $i'j''$ and $i''j'$ are properly colored.



The categorical graph product: $C_n \times K_2$. **Edge.**

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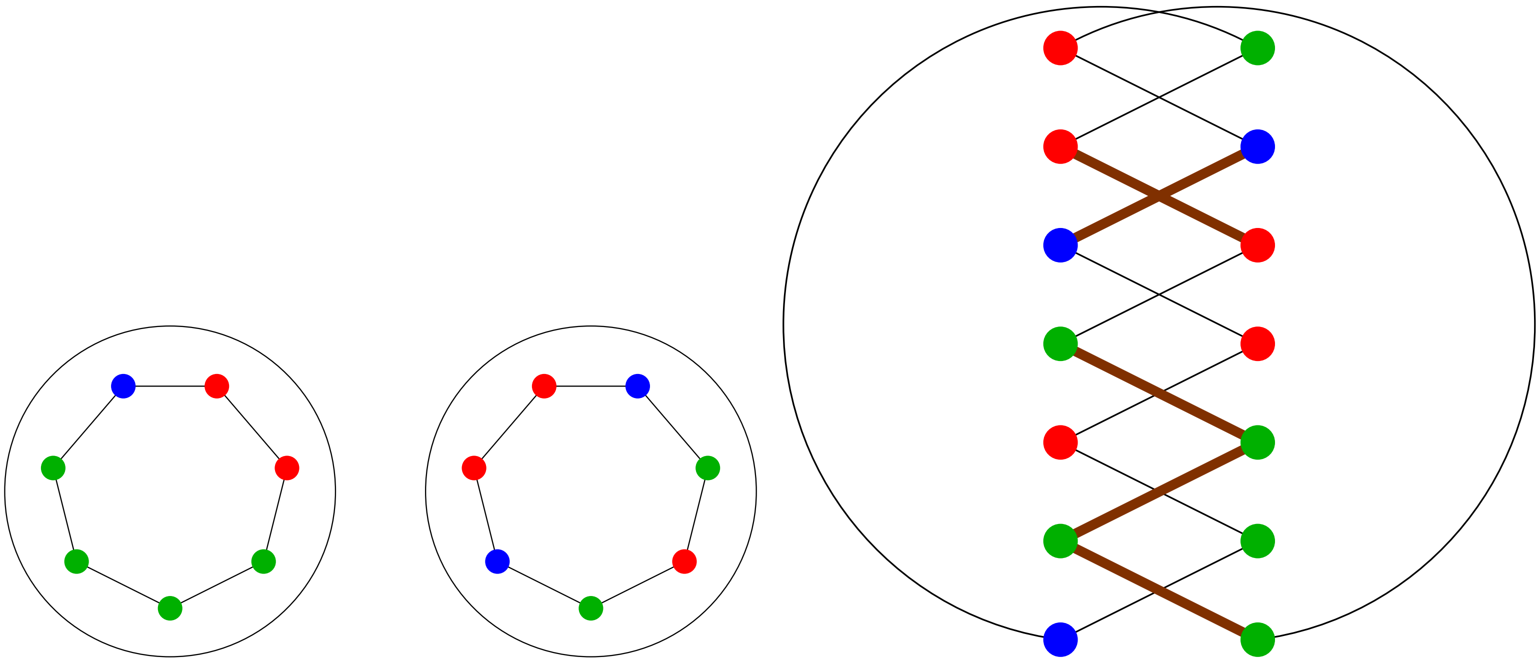

The categorical graph product: $C_n \times K_2$. **No Edge.**

The 3-Coloring Exponential Graph K

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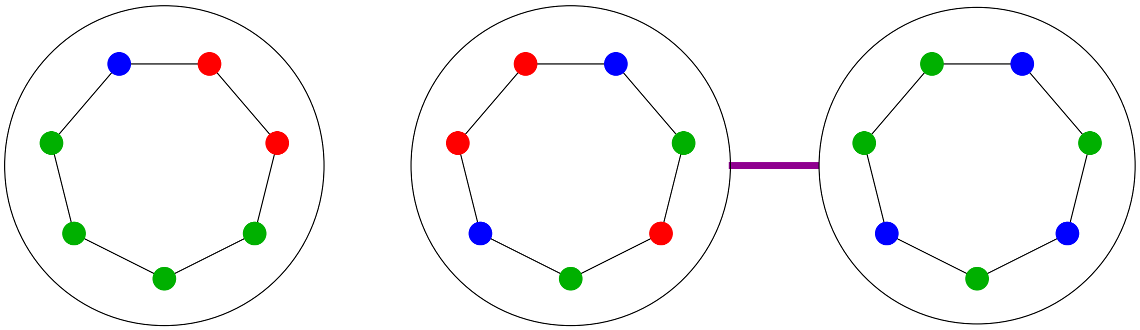
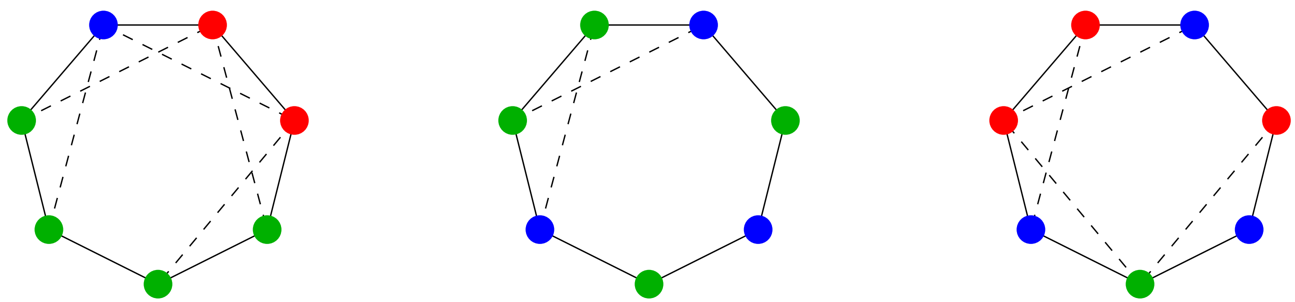


- Add an edge between two 3-colorings if the bipartite graph between them is properly colored.
- K is not bipartite, but we now describe an induced subgraph that is.

Definition of Fixed Points

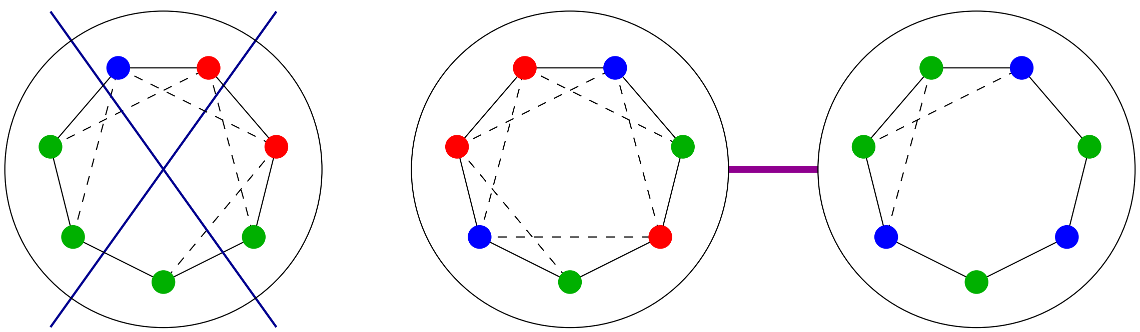


A “fixed point” means color of that node in a neighboring vertex is “fixed”.

Induced Subgraph that is Bipartite (First Rule)

- Vertices of $K = K_3^{C_n}$ are all 3-colorings of C_n.

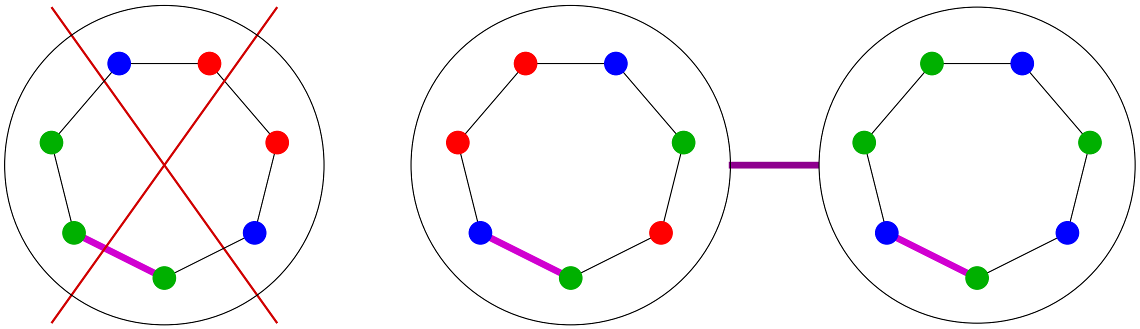
(1) Keep vertices with an even number of fixed points.
(Call this the *even component* of K.)



Induced Subgraph that is Bipartite (Second Rule)

- Vertices of $K = K_3^{C_n}$ are all (i.e. 3^n) 3-colorings of C_n.

(1) Keep vertices with an even number of fixed points.



(2) Fix any edge e in C_n and remove all vertices of K corresponding to colored copies of C_n in which edge e is monochromatic.

- Call this graph K_e.

- Theorem: K_e is bipartite [El-Zahar and Sauer 1985].

Application of K_e Being a Bipartite Graph

We can 3-color the even component of $K_3^{C_n}$.

Why do we want to do this ?

When $\chi(H) > k$, we can k-color K_k^H iff Hedetniemi's Conjecture is true [El-Zahar and Sauer 1985].

They also showed:

The problem of 3-coloring K_3^H when $\chi(H) > 3$ can be reduced to:

The problem of 3-coloring the even component of $K_3^{C_n}$.

Application of K_e Being a Bipartite Graph

Goal: 3-color the even component of $K_3^{C_n}$.

Algorithm: [Tardif 2006]

Let $e = ab$.

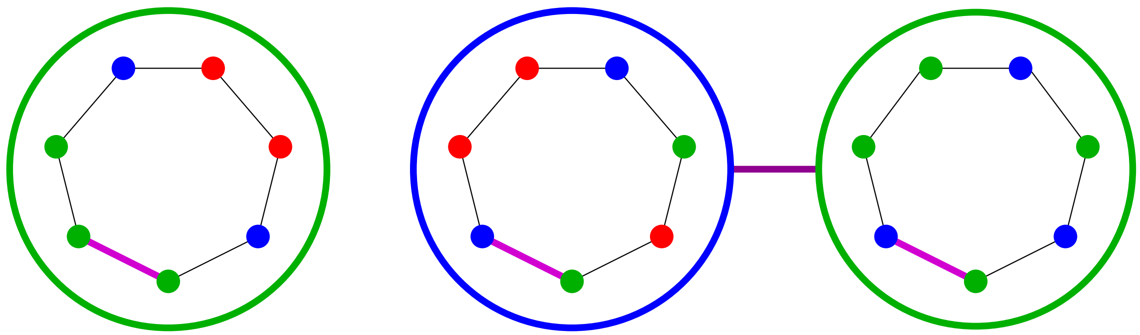
Find bipartition of K_e.

If vertex in K_e belongs to LEFT side, color is $color(a)$.

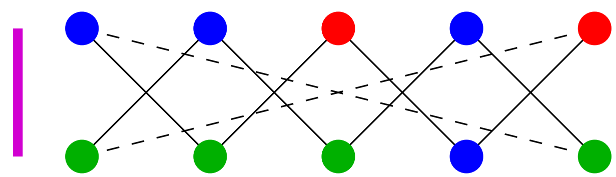
If vertex in K_e belongs to RIGHT side, color is $color(b)$.

If vertex has $c = color(a) = color(b)$, then color is c.

3-Coloring the Even Component of $K_3^{C_n}$.



For any edge e, copies of C_n in which e is monochromatic form hitting set for the odd cycles in the even component of $K_3^{C_n}$.



Tardif's Question on Explicit Colorings

Is there is an "explicit bipartition" for K_e ? [Tardif 2006]

Problem: Given a 3-colored copy of C_n such that:

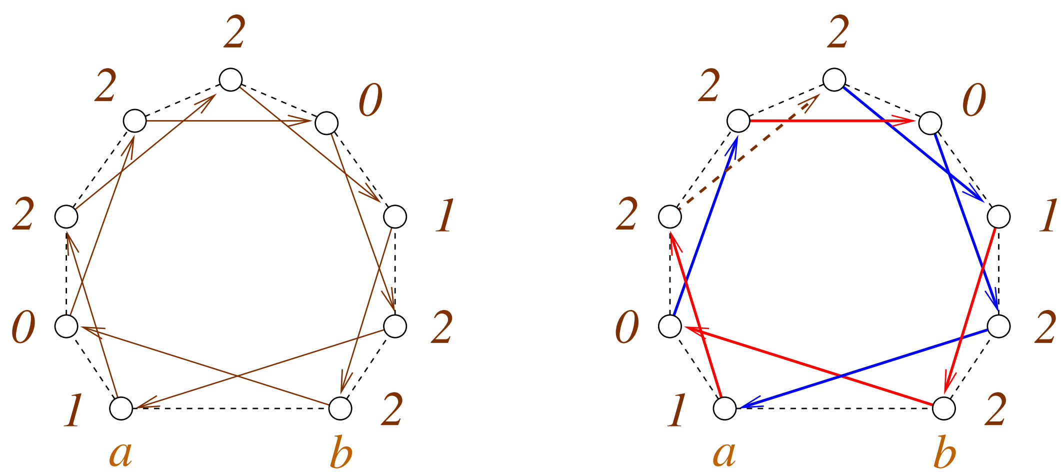
1. there is an even number of fixed points, and
2. e is not monochromatic,

Assign this copy to one of the sides of the bipartition in time poly(n).

An Explicit Bipartition for K_e

Compute the *label* ℓ of a vertex: $\{01, 12, 20\} = +1, \{10, 21, 02\} = -1$.

Suppose label is sum of all arcs in "chord cycle". Here, $\ell = 0$.



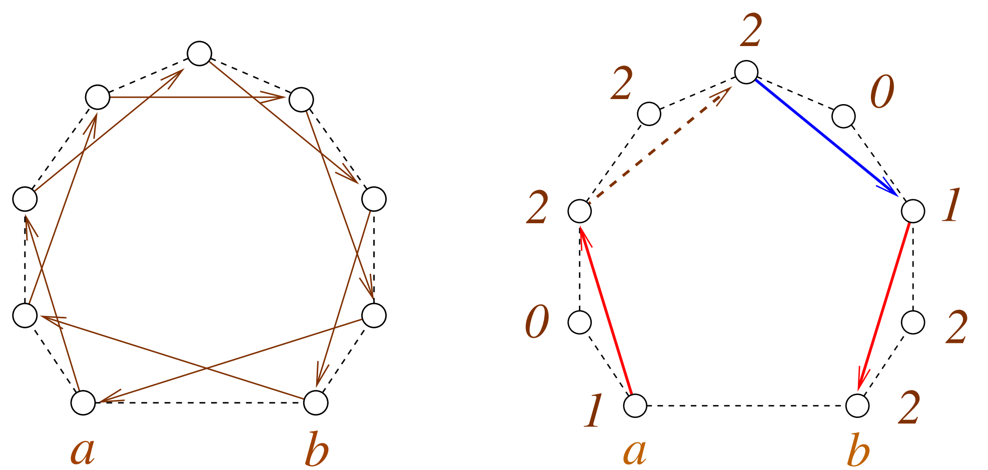
All vertices in the same connected component have same label.

An Explicit Bipartition for K_e

Compute the *label* ℓ of a vertex: $\{01, 12, 20\} = +1, \{10, 21, 02\} = -1$.

Suppose label $\ell = 0$. Recall $e = ab$.

Then "petit chemin" from a to b is either majority **Red** or majority **Blue**.



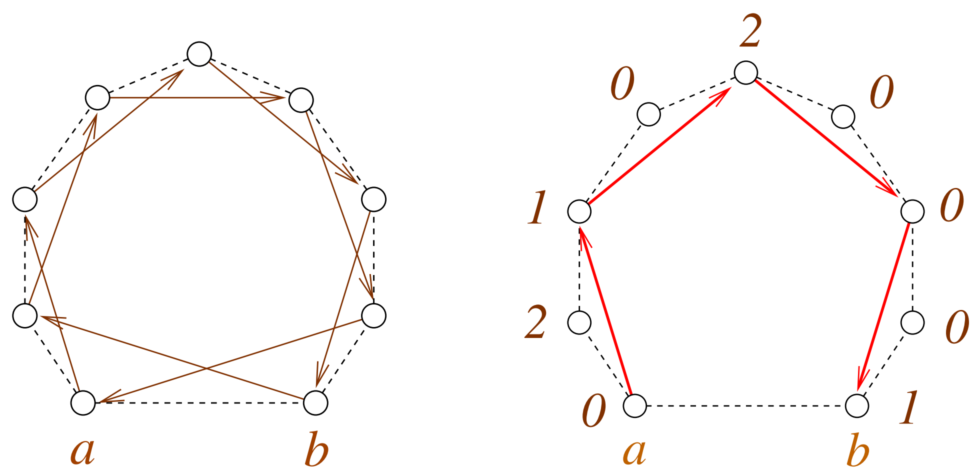
All vertices in the same component have same label.

An Explicit Bipartition for K_e

Compute the *label* ℓ of a vertex: $\{01, 12, 20\} = +1, \{10, 21, 02\} = -1$.

Suppose label $\ell > 0$ (e.g., $\ell = 6$).

Then "petit chemin" from a to b is either $> \frac{\ell}{2}$ or $< \frac{\ell}{2}$.



All vertices in the same component have same label.

An Explicit Bipartition for K_e

Rule:

1. If $p(a,b) > \ell/2$, LARGE side.
2. If $p(a,b) < \ell/2$, SMALL side.

Main challenge: show that two neighbors f and g have $p_f(a,b)$ and $p_g(a,b)$ values that are anti-correlated:

$$\ell - 1 \leq p_f(a,b) + p_g(a,b) \leq \ell + 1.$$

Proof idea: Prove by induction that $p_f(a,b)$ and $P_g(b,a)$ are correlated.

Conclusions + Open Question

Assume $\chi(H) \geq 4$ and let $f(H)$ denote a 3-colored copy of H.

If $f(H)$ is a vertex of a 3-chromatic component of K_3^H, we can assign a color to this vertex in time $O(|H|)$.

If $f(H)$ is an isolated vertex of K_3^H, we can determine this in time $O(|H|)$.

If $f(H)$ is a vertex of a bipartite component of K_3^H, then we can assign a color in time $O(|H|) \cdot W$, where W is the number of vertices in bipartite components that we have seen so far (i.e., time is "input sensitive").

Question: Given 3-colored copy of H, determine if this copy contains an odd cycle with an even number of fixed points.