Fractional chromatic number and girth

François Pirot, joint work with J.S. Sereni

JGA 2018

Colouring and Stable Sets

Stable sets

A *stable set* of a graph is a subset of its vertices with no edges in between. It can be:

- *maximal*: Any other vertex of the graph shares an edge with at least one of its vertices.
- *maximum*: Its size is the biggest possible in the graph.

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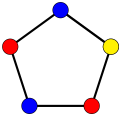
Chromatic Number

- A *k-colouring* of some graph $G = (V, E)$ is a function $c : V \to [k]$ such that $uv \in E \Rightarrow c(u) \neq c(v)$. In other words, this is a partition of V of size k into stables sets. The *chromatic number* $\chi(G)$ of G is the minimum k such that there is a k-colouring of G.

A colouring of C_5 needs 3 colours : $\chi(C_5) = 3$



The chromatic number formulated as a Linear Program

The Linear Program

Let G be some graph, and $\mathcal{S}$ be the set of (maximal) stable sets of G.

$$\chi(G) = \min \sum_{S \in \mathcal{S}} w_S$$

$$\text{such that} \quad \begin{cases} \forall v \in V(G), & \sum_{S \ni v} w_S \geq 1 \\ \forall S \in \mathcal{S}, & w_S \in \{0, 1\} \end{cases}$$

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Fractional Chromatic Number

The *fractional chromatic number* $\chi_f(G)$ of some graph G is the solution to the fractional relaxation of the linear program computing $\chi(G)$.
It can also be defined through *fractional colourings*.

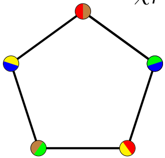
- A (a,b)-*colouring* of some graph $G = (V, E)$ is a function $c : V \to \binom{[a]}{b}$ such that $uv \in E \Rightarrow c(u) \cap c(v) = \varnothing$.
- $\chi_f(G) = \inf \left\{ \frac{a}{b} \;\middle|\; G \text{ has a } (a,b)\text{-colouring} \right\}$

Remark

A (a, b)-colouring of some graph G can be seen as a a-colouring of $G \boxtimes K_b$.

A $(5, 2)$-colouring of C_5 (every maximal stable set gets weight $\frac{1}{2}$)

$$\chi_f(C_5) = 2.5$$



$\Leftrightarrow$

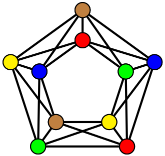


Illustration and Properties

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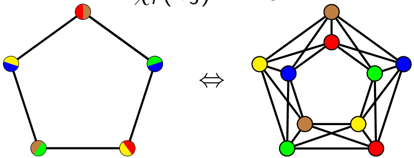

Properties

For any graph G on n vertices, of maximum degree Δ, clique and stability numbers ω and α (size of the biggest clique / stable set):

$$\omega \leq \chi_f(G) \leq \chi(G) \leq \Delta + 1$$

$$\frac{n}{\alpha} \leq \chi_f(G)$$

Known results

Theorem (Molloy, 2017)

If a graph G of maximum degree Δ is triangle-free (so of girth at least 4), then:

$$\chi(G) \leq (1 + o(1))\frac{\Delta}{\ln \Delta}.$$

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For any graph G of maximum degree Δ and of clique number ω,

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Comment on these theorems

The second theorem gives a bound for triangle-free graph: $\chi_f \leq \frac{\Delta+3}{2}$. This is a lot weaker than the bound given by the first theorem in the asymptotic, but for small degrees ($\Delta \geq 4$) there is no better known bound!

The greedy fractional colouring algorithm

For some graph G, fix a probability distribution on the stable sets of any induced subgraph of G. We denote by $\mathbf{S}$ the corresponding random stable set.

The algorithm

1. Set $G_0 = G$.
2. At each step i, give to every stable set I of G_i a weight proportional to $\mathbb{P}[\mathbf{S} = I]$ in such a way that the new maximum induced weight on the vertices of G_i is exactly 1. (We take care never to exceed 1)
3. Define G_{i+1} as the graph G_i where those new vertices with weight 1 are removed. Stop if this is the empty graph.

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Lemma (weak version)

If for every induced subgraph H of G, every random stable set $\mathbf{S}$ of H, and every vertex $v \in V(H)$,

$$\alpha \mathbb{P}[v \in \mathbf{S}] + \beta \mathbb{E}[|N(v) \cap \mathbf{S}|] \geq 1,$$

then the returned fractional colouring is of weight at most

$$\alpha + \beta \Delta(G).$$

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Lemma (P, Sereni [2018+])

Fix some depth $r \geq 1$. If for every induced subgraph H of G, every random stable set $\mathbf{S}$ of H, and every vertex $v \in V(H)$,

$$\sum_{i=0}^{r} \alpha_i(v) \mathbb{E}\left[|N^i(v) \cap \mathbf{S}|\right] \geq 1,$$

then the returned fractional colouring is of weight at most

$$\max_{v \in V(G)} \sum_{i=1}^{r} \alpha_i(v) |N^i(v)|.$$

Proof of the fractional Reed's bound

- Let v be some vertex in G, and $\mathbf{S}$ a random maximum stable set of G. We show that

$$\frac{\omega(G)+1}{2}\mathbb{P}\left[v \in \mathbf{S}\right] + \frac{1}{2}\mathbb{E}\left[|N(v) \cap \mathbf{S}|\right] \geq 1$$

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  1. Event X_1: The uncovered part of $N(v)$ is a clique (of size $k \in [1, \omega]$). Then the events $u \in \mathbf{S}$ for $u \in N[v] \setminus N(\mathbf{R})$ are equally likely.

$$\mathbb{P}\left[v \in \mathbf{S} \mid X_1\right] = \frac{1}{k}, \quad \mathbb{E}\left[|N(v) \cap \mathbf{S}| \mid X_1\right] = \frac{k-1}{k}$$

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  2. Event X_2: At least 2 independent neighbours of v are not covered by $\mathbf{R}$. Then $|N(v) \cap \mathbf{R}| \geq 2$, and $v \notin \mathbf{R}$.

$$\mathbb{P}\left[v \in \mathbf{S} \mid X_2\right] = 0, \quad \mathbb{E}\left[|N(v) \cap \mathbf{S}| \mid X_2\right] \geq 2$$

New results

Limitation

When using the uniform distribution on maximum stable sets, the algorithms cannot beat Reed's bound for some bipartite graphs. These are highly irregular; considering regular graphs helps avoiding this, at the cost of proving bounds only on n/α instead of χ_f.

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Independence ratio upper bounds (P, Sereni [2018+])

We propose some upper bounds for $\frac{n}{\alpha}$ for graphs G on n vertices and stability number α, depending on its girth and (small) maximum degree.

girth	4	5	6	8	10	12
$\Delta = 3$	2.8	2.8	≈ 2.7272	≈ 2.6252	≈ 2.5571	≈ 2.5103
$\Delta = 4$	3.375	3.28	≈ 3.1538	≈ 3.0385	?	?

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Corollary

These are upper bounds for χ_f in the class of vertex transitive graphs.

An insight of the method

Less constrained lemma (P, Sereni [2018+])

Fix some depth $r \geq 1$, and let S be a random stable set of some d-regular graph G, such that for every vertex $v \in V(G)$,

$$\sum_{i=0}^{r} \alpha_i \mathbb{E}\left[|X_i(v) \cap S|\right] \geq 1,$$

where $X_i(v)$ is the random variable counting the number of paths of length i beginning in v and ending in S. Then

$$\frac{n}{\alpha(G)} \leq \alpha_0 + \sum_{i=1}^{r} \alpha_i d(d-1)^{i-1}.$$

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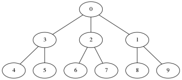
Method overview

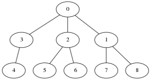
- Enumerate all the possible partially covered neighbourhoods up to depth r of G, and compute $(\mathbb{E}_{X_0}, \ldots, \mathbb{E}_{X_r})$ for each of them.
- Compute the best possible $(\alpha_0, \ldots, \alpha_r)$ through a linear program.

Computer work

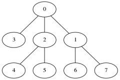
The 10 possible partially covered neighbourhoods of depth 2 for 3-regular graphs of girth 6, with the corresponding value for $(\mathbb{E}_{X_0}, \mathbb{E}_{X_1}, \mathbb{E}_{X_2})$.



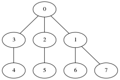
$(1, 0, 6)$



$(1, 0, 5)$



$\left(\frac{1}{2}, \frac{1}{2}, 4\right)$



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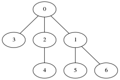
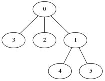
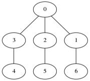


$\left(\frac{1}{3}, 1, \frac{8}{3}\right)$



$(0, 2, 2)$



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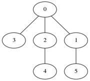
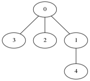


$\left(\frac{1}{5}, \frac{8}{5}, \frac{6}{5}\right)$



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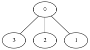

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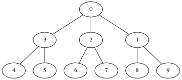

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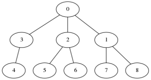

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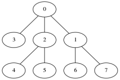

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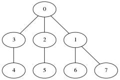

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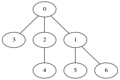

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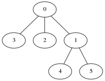

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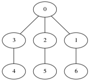

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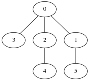

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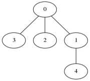

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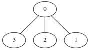

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Linear Program computing the desired bound for d-regular graphs

minimize $\quad \alpha_0 + \sum_{i=1}^{r} d(d-1)^{i-1}\alpha_i$

such that
$$\begin{cases} \forall v, \forall R, & \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[X_i(v) \mid S \setminus N^r[v] = R\right] \geq 1 \\ \forall i, & \alpha_i \geq 0 \end{cases}$$

A better than Reed's upper bound for graphs of girth 7

Theorem (P, Sereni [2018+])

For any graph G of maximum degree Δ and girth at least 7:

$$\chi_f(G) \leq \frac{2\Delta + 9}{5}$$

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Comment

This result was obtained with neighbourhoods at depth $r = 2$.
Surprisingly, increasing the depth does not give any better bound for cubic graphs, even though we know that the real bound is 2.8 and not the value 3 given by the theorem.
This is enough to believe that the value 3.4 given by the theorem for graphs of maximum degree 4 and girth 7 is far from the optimal.

Presentation of the tools needed for the proof

Hard-core distribution on maximal stable sets

We use a *hard-core* random distribution on *maximal* stable sets instead of the *uniform* distribution on *maximum* stable sets.
Let $\mathbf{S}$ be the corresponding random stable set, and $\mathcal{S}$ be the set of maximal stable sets of G, then

$$\forall I \in \mathcal{S}, \mathbb{P}_\lambda[\mathbf{S} = I] = \frac{\lambda^{|I|}}{\sum_{J \in \mathcal{S}} \lambda^{|J|}}, \text{ for any } \lambda > 0.$$

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Main difficulty

For some neighbourhood $W = N^r[v]$, let $\mathbf{R} := \mathbf{S} \backslash W$. It must hold that the vertices of $\overline{W}$ not covered by $\mathbf{R}$ must be covered by $\mathbf{S} \cap W$.
We consider neighbourhoods at distance $r = 2$, and ask for girth 7 in order to ensure that the vertices in $\overline{W}$ can be covered by at most one vertex in W.

Going further

- We can imagine replacing the girth constraint by any other local constraint which is well expressed when we enumerate neighbourhoods at depth r: for instance bounding the number of triangles in which every vertex appears, or forbidding any fixed small enough subgraph.

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Thank you for your attention!