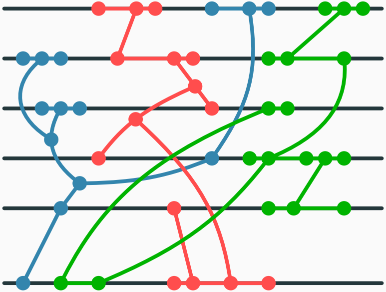
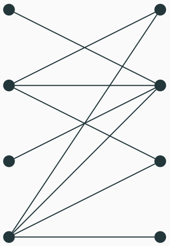
Packing and covering minors

Jean-Florent Raymond
(TU Berlin)

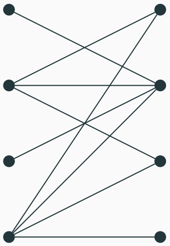


Joint work with Wouter Cames van Batenburg, Tony Huynh, and Gwenaël Joret (Université Libre de Bruxelles).

Packing and covering in bipartite graphs



Max. number
of disjoint edges?



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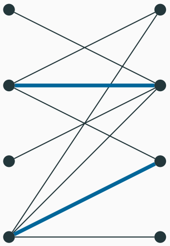

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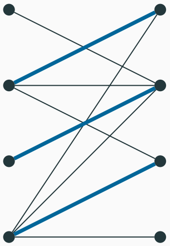

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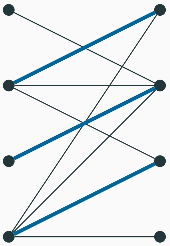

$\text{pack}_{K_2} = 3$

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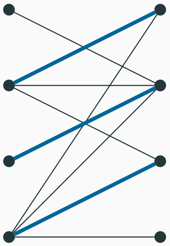

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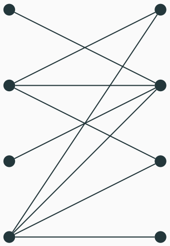
Min. number of vertices to cover all edges?



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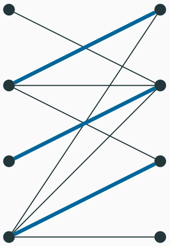

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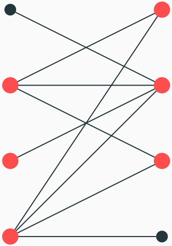

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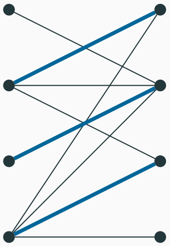

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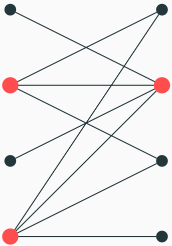

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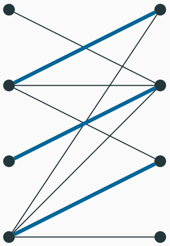

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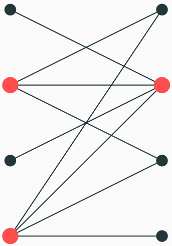
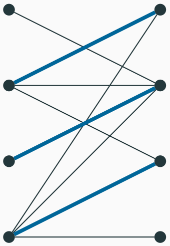


$\text{cover}_{K_2} = 3$

Max. number of disjoint edges?



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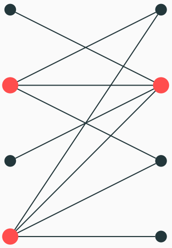
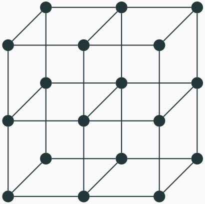


$\text{cover}_{K_2} = 3$

$\text{cover} = \text{pack}$

(Kőnig's Theorem, 1931)

Packing and covering cycles



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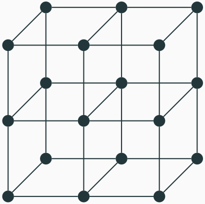

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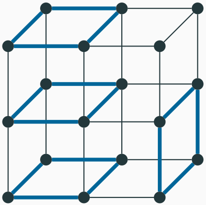

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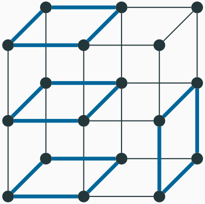

$\text{pack}_{\text{cycles}} = 4$

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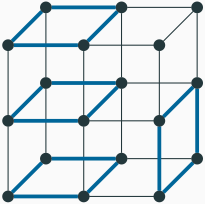

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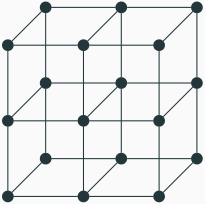
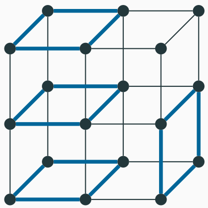


Max. number of disjoint cycles?

Min. number of vertices to cover all cycles?



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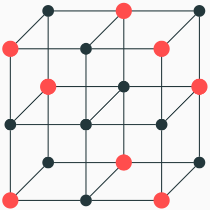

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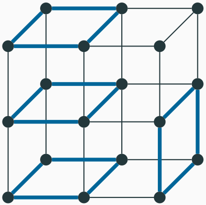

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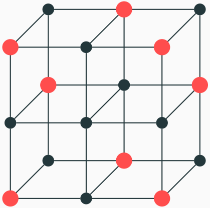
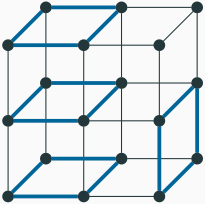
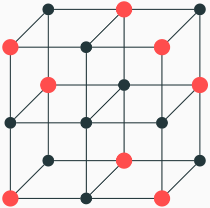


$\text{cover}_{\text{cycles}} = 8$

Max. number of disjoint cycles?

Min. number of vertices to cover all cycles?





$\text{pack}_{\text{cycles}} = 4$

$\text{cover}_{\text{cycles}} = 8$

$\text{pack} \leqslant \text{cover} \leqslant c \cdot \text{pack} \log \text{pack}$

(Erdős-Pósa Theorem, 1965)

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Every graph has one of the following:

- k *vertex-disjoint cycles;*
- *a feedback vertex set of size* $O(k \log k)$.

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Min-max theorem (like Kőnig's and Menger's theorems, etc.).

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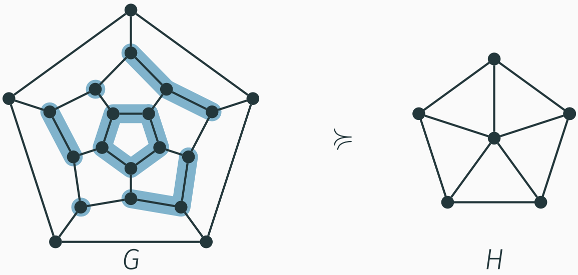
Our goal: generalize from cycles to minor-models.

Definition

An *H-model* in G is a set $\{S_u\}_{u \in V(H)}$ of disjoint subsets of $V(G)$ s.t.

- the $G[S_u]$'s are connected;
- edge uv in $H \Rightarrow$ edge between S_u and S_v in G.



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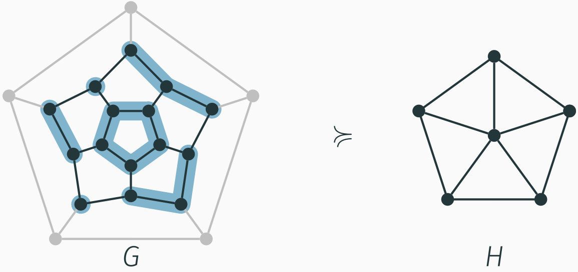

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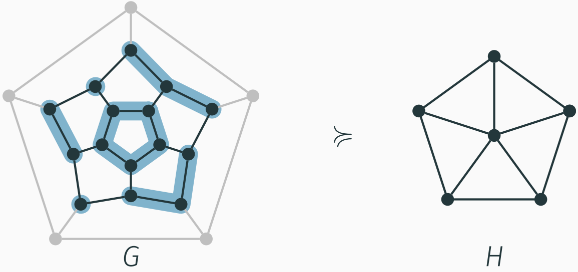

G has a H-model $\iff$ H is a minor of G

Definition

H has the Erdős-Pósa property if there is a function f s.t., for every graph G and $k \in \mathbb{N}$,

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- there is $X \subseteq V(G)$ s.t. $G - X$ is H-minor free and $|X| \leqslant f(k)$.

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Theorem (Robertson and Seymour, 1986)

H has the Erdős-Pósa property iff H is planar.

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With which gap?

Graph H	EP gap	Reference	
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A non-exhaustive history of Erdős-Pósa gaps

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Best possible:

- H not planar $\Rightarrow$ no Erdős-Pósa property;

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Best possible:

- H not planar $\Rightarrow$ no Erdős-Pósa property;
- H has a cycle $\Rightarrow$ no $o(k \log k)$ gap.

The key lemma

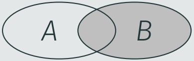
"every graph has a *small* H-model or a large *useless* part" "every graph has a *small* H-model or a large *useless* part"

Lemma (Cames van Batenburg, Huynh, Joret, R., 2018+)

For every graph G and every planar graph H,

- *G has an H-model of size $O(\log |G|)$;*

  or

- $G =$ 

  $G[B]$ *is H-minor free*
  $|B| \geqslant \text{large}(|A \cap B|)$

Proof sketch for $H = K_3$

Goal: "G has a *small* K_3-model or a large *useless* part"

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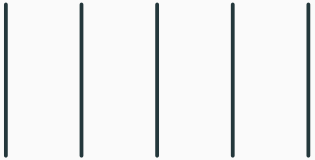
Maximum collection of disjoint paths of length ℓ:

(covering G, for simplicity)



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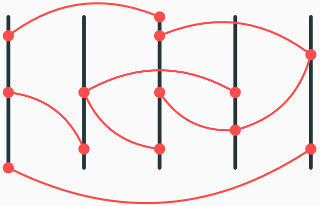

- either every path sees $\geqslant 3$ other paths

Goal: "G has a *small* K_3-model or a large *useless* part"

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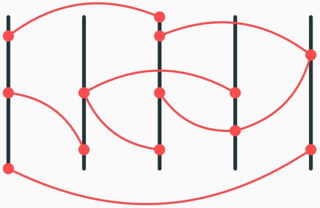

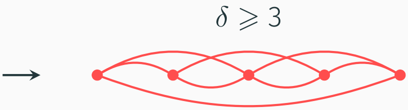

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Proof sketch for $H = K_3$

Goal: *"G has a small K_3-model or a large useless part"*

Maximum collection of disjoint paths of length ℓ:
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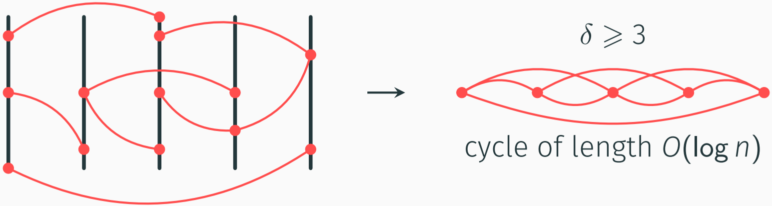

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Proof sketch for $H = K_3$

Goal: "*G* has a *small* K_3-model or a large *useless* part"

Maximum collection of disjoint paths of length ℓ:

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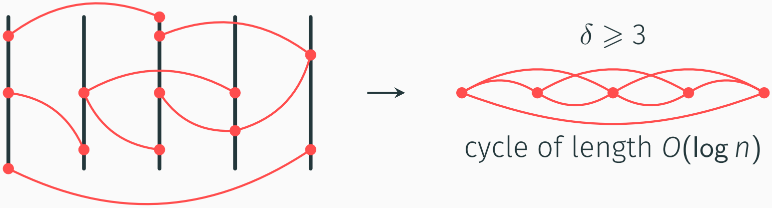

- either every path sees $\geqslant 3$ other paths:

cycle of length $O(\ell \cdot \log |G|)$

Proof sketch for $H = K_3$

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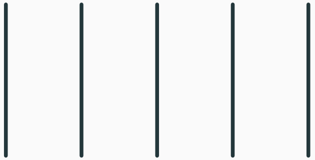

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Maximum collection of disjoint paths of length ℓ:

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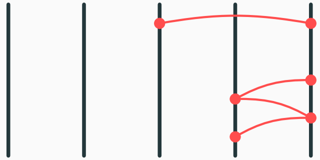

- either every path sees $\geqslant 3$ other paths:

  cycle of length $O(\ell \cdot \log |G|)$

- or one path sees $\leqslant 2$ other paths

Proof sketch for $H = K_3$

Goal: "*G has a small K_3-model or a large useless part*"

Maximum collection of disjoint paths of length ℓ:
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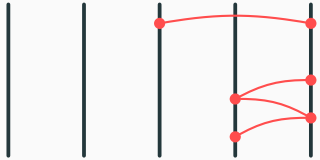

cycle of length $\leqslant 2\ell$

- either every path sees $\geqslant 3$ other paths:
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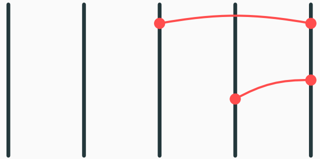

there are $\leqslant 2$ incident edges

- either every path sees $\geqslant 3$ other paths:
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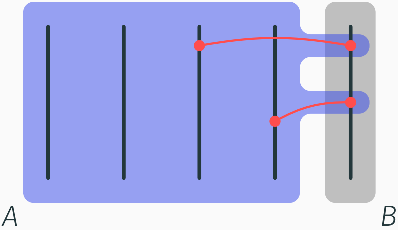

- B is K_3-minor free
- $|B| \geqslant \text{large}(|A \cap B|)$

- either every path sees $\geqslant 3$ other paths:
  cycle of length $O(\ell \cdot \log |G|)$
- or one path sees $\leqslant 2$ other paths

Proof sketch for $H = K_3$

Goal: *"G has a small K_3-model or a large useless part"*

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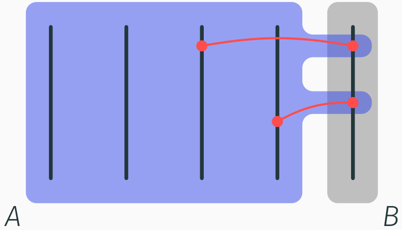

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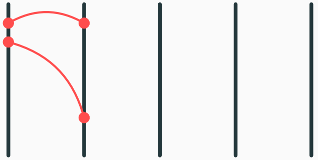
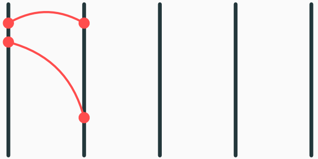
  cycle of length $\leqslant 2\ell$ or large useless part.

How to generalize?



Crucial property: we can conclude when two paths are connected with many edges.



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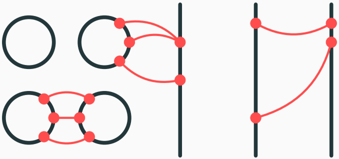
Possible extension to $H = K_4$:



Pack cycles of bounded size first, then paths.

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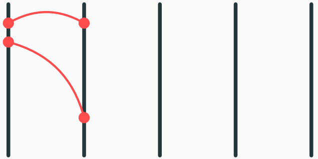

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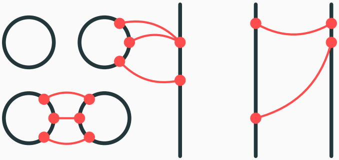
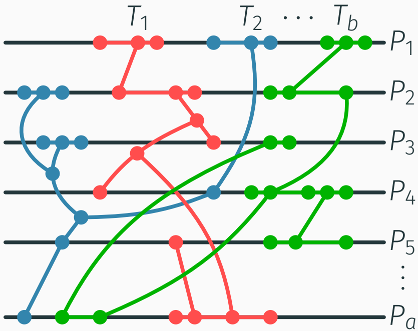


Pack cycles of bounded size first, then paths.

$\rightsquigarrow$ gap $O(k \log k)$ when H is a wheel
(Aboulker, Fiorini, Huynh, Joret, R. and Sau, 2018)

Orchards

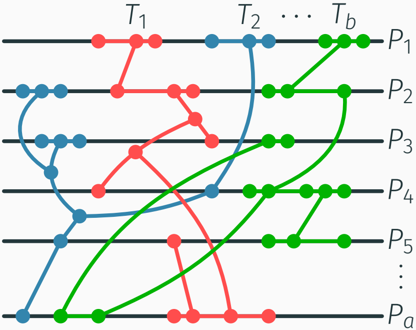


An $a \times b$-orchard in G consists in collections

- $P_1, \dots, P_a$ of vertex-disjoint (horizontal) paths; and
- $T_1, \dots, T_b$ of vertex-disjoint (vertical) trees,

s.t. for every $i \in [a]$, $j \in [b]$:

- $P_i \cap T_j \neq \emptyset$ and connected; and
- each leaf of T_j lies on some horizontal path.



Consequences

Consequence 1/4: Algorithms

$\text{pack}_H(G)$ max. number of disjoint H-models in G
$\text{cover}_H(G)$ min. size of a cover of H-models in G

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Theorem (from our results)

For every planar graph H, there is a polytime $O(\log(\text{OPT}))$-approximation algorithm for pack_H.

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(idem for cover_H, but $O(1)$-approximations are already known)

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For every $m \in \mathbb{N}_{\geqslant 1}$ there is a function f s.t., for every $k \in \mathbb{N}$ and every graph G,

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Consequence 3/4: Packing cycles with modularity constraints

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- from our result: $f(k) = O(k \log k)$ (tight)

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The previous theorem also follows from our results.

Open problems

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Thank you for your attention!