List-coloring claw-free graphs with small clique number

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Abstract

Chudnovsky and Seymour proved that every connected claw-free graph that contains a stable set of size 3 has chromatic number at most twice its clique number. We improve this for small clique size, showing that every claw-free graph with clique number at most 3 is 4-choosable and every claw-free graph with clique number at most 4 is 7-choosable. These bounds are tight.

Keywords: Claw-free graphs, chromatic number, list coloring.

1 Introduction

We study the relation between the clique number $\omega(G)$ and the chromatic number $\chi(G)$ for graphs G that are claw-free, i.e., that do not contain a star on four vertices as an induced subgraph. In general, χ is not upper-bounded by any function of ω, but the second author observed that for every claw-free graph G, the maximum degree $\Delta(G)$ is bounded by a function of $\omega(G)$ [6]. It follows that for every claw-free graph not only the chromatic number but

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also the list chromatic number is bounded by a function of the clique number. The *list chromatic number* of a graph G, denoted by $ch(G)$, is the least k such that if every vertex is given a list of at least k colors, G can be properly colored so that every vertex receives a color from its own list.

It was conjectured by Gravier and the third author [4] that every claw-free graph G satisfies $ch(G) = \chi(G)$. This conjecture generalizes the famous list coloring conjecture, stating that χ and ch coincide for line-graphs of loopless multigraphs.

Chudnovsky and Seymour [2] proved that every connected claw-free graph G that has a stable set of size 3 satisfies $\chi(G) \leq 2\omega(G)$. In fact, they strengthen this to all induced subgraphs of such graphs, which they call *tame graphs*. They claim their bound is best possible, however our results show this is not the case for $\omega = 3, 4$. The explanation is that the graphs G'_n defined in Section 4 of [2] satisfy $\omega(G'_n) = n + 1$ for $n \geq 3$ (instead of the claimed $\omega(G'_n) = n$). Thus $\chi(G'_n) = 2\omega(G'_n) - 2$ holds for their example, showing that their result is at most 2 away from best possible. It is also proved in [2] that if further technical conditions are added, then also $ch(G) \leq 2\omega(G)$.

The upper bound 2ω on χ and ch in [2] is a consequence of a deep decomposition theorem for connected claw-free graphs having a stable set of size 3. Unfortunately, graphs with no stable set of size 3 have no nice decomposition, and their structure dictates the general behavior of χ and ch with respect to ω in the whole class of claw-free graphs, as we will see below. In fact, as Table 1 shows, the bound 2ω on χ fails for $\omega \geq 7$.

Our aim here is to improve the bound of $\chi \leq 2\omega$ for small values of ω but for all claw-free graphs (that is, without using any decomposition theorem) and to show that the better bounds hold in the wider context of list coloring. For each integer k, let $\mathcal{C}_k$ be the class of claw-free graphs that contain no clique of size k. Let

$$\chi(\mathcal{C}_k) = \max\{\chi(G) \mid G \in \mathcal{C}_k\} \text{ and } ch(\mathcal{C}_k) = \max\{ch(G) \mid G \in \mathcal{C}_k\}.$$

Our purpose is to determine the exact values of $\chi(\mathcal{C}_k)$ and $ch(\mathcal{C}_k)$ for small values of k.

Ramsey [11] proved that for any two integers $k \geq 1$ and $\ell \geq 1$ there exists an integer N such that every graph with N vertices contains either a clique of size k or a stable set of size ℓ. The smallest such integer N is denoted by $R(k, \ell)$. Consequently, there exists at least one graph on $R(k, \ell) - 1$ vertices that contains no clique of size k and no stable set of size ℓ; any such graph is called a *Ramsey* graph. The exact value of $R(k, \ell)$, and the corresponding Ramsey graphs, are known only for small values of k and ℓ. We will be interested in $R(k, 3)$, for the following reasons. Suppose that G is a graph with no clique of size k and no stable set of size 3. Then G is claw-free. Moreover, since G has no stable set of size 3, its chromatic number is at least $|V(G)|/2$. This holds in particular when G is any Ramsey graph that corresponds to the pair $(k, 3)$. It follows that:

$$\chi(\mathcal{C}_k) \geq \left\lceil \frac{R(k,3)-1}{2} \right\rceil. \tag{1}$$

Reed [12] conjectured that every graph G satisfies $\chi(G) \leq \lceil \frac{\Delta(G)+\omega(G)+1}{2} \rceil$. King [9] proved this conjecture for claw-free graphs. Note that if G is any claw-free graph with no clique of size k, then the subgraph of G induced by the neighborhood of any vertex x contains no clique of size $k-1$ and no stable set of size 3, so the degree of every vertex is at most $R(k-1, 3) - 1$. Hence King's result implies that:

$$\chi(\mathcal{C}_k) \leq \left\lceil \frac{R(k-1,3)+k}{2} \right\rceil. \tag{2}$$

Inequalities (1) and (2) are illustrated in the table below, based on the known value of $R(k, 3)$ for small k (see [10]). It follows from (1) together with the upper bound on Δ and [1, 8] that there exist constants c and c' such that $ck^2/\log k \leq \chi(\mathcal{C}_k) \leq ch(\mathcal{C}_k) \leq c'k^2/\log k$. If the conjecture in [4] is true, the last three lines in the table should be equal.

k	3	4	5	6	7	8	9	10
$R(k,3)$	6	9	14	18	23	28	36	40–43
$\chi(\mathcal{C}_k) \geq$	3	4	7	9	11	14	18	20
$\chi(\mathcal{C}_k) \leq$	3	5	7	10	13	16	19	23
$ch(\mathcal{C}_k)$								

Table 1: Ramsey numbers and bounds on $\chi(\mathcal{C}_k)$.

It is easy to see that every graph G in $\mathcal{C}_3$ is the disjoint union of cycles of length at least 4 and paths, and, consequently, G satisfies either $\chi(G) =$

$ch(G) = 1$ (if G has no edge) or $\chi(G) = ch(G) = 2$ (if G has an edge and no odd cycle) or $\chi(G) = ch(G) = 3$ (if G contains an odd cycle). This establishes the equality $ch(\mathcal{C}_3) = 3$ in the first empty box of the table. We shall now establish the exact value of $\chi(\mathcal{C}_k)$ and $ch(\mathcal{C}_k)$ for the next two values of k:

$$\chi(\mathcal{C}_4) = ch(\mathcal{C}_4) = 4$$

and

$$\chi(\mathcal{C}_5) = ch(\mathcal{C}_5) = 7.$$

The lower bounds for classes $\mathcal{C}_4$ and $\mathcal{C}_5$ in Table 1 are based on Ramsey graphs with no stable set of size 3. However, $\chi(\mathcal{C}_4) \geq 4$ can also be demonstrated by a connected claw-free graph with a stable set of size 3, for example by the graph that consists in a 5-wheel with an additional vertex joined to two consecutive vertices on the 5-cycle. We do not know whether $\chi(\mathcal{C}_5) \geq 7$ can be demonstrated by a tame graph or, on the contrary, every tame graph G with $\omega(G) = 4$ satisfies $\chi(G) \leq 6$. This would be best possible since [2] exhibits tame graphs with $\chi(G) = 2\omega(G) - 2$; for example, the graph called G'_3 in [2, p. 569] is tame and satisfies $\omega(G'_3) = 4$ and $\chi(G'_3) = 6$.

The proofs of the upper bounds are given in Sections 3 and 4. Section 2 provides useful tools that will be used repeatedly in the proofs of the two main results.

2 List-coloring

Given a list assignment L of colors to the vertices of a graph G, an *L-coloring* of G is a proper coloring of G such that every vertex v has a color from $L(v)$. If such a coloring exists, G is said to be *L-colorable*. A graph G is *k-choosable* if, for every list assignment L to the vertices of G such that $|L(v)| \geq k$ for every vertex v, G is L-colorable. The following result, due to Erdős, Rubin and Taylor [3], can be seen as an analogue of Brooks theorem for list-coloring.

Theorem 2.1 ([3]). *Let G be a graph and, for each vertex v of G, let $L(v)$ be a list of allowed colors with $|L(v)| \geq d_G(v)$. Then either G has an L-coloring, or every block of G is a clique or an odd cycle.*

A *diamond* is a graph obtained from a clique of size 4 by removing an edge. A *triangle* is a clique of size 3.

Lemma 2.2. *Let G be a diamond and L be a list assignment on $V(G)$ such that $|L(v)| \geq 2$ for every vertex $v \in V(G)$. Then G is L-colorable unless the three lists on the vertices of some triangle of G have size 2 and are identical.*

Proof. Let v_1, v_3 be the vertices of degree 3 and v_2, v_4 be the vertices of degree 2. Let $L_i = L(v_i)$ for each $i \in \{1, \dots, 4\}$. We assume that among v_1, v_2, v_3 (resp. v_1, v_3, v_4), two vertices have non-identical lists.

If $L_1 \neq L_3$ there are at least three sets $\{a, b\}$ with $a \neq b$ and $(a, b) \in L_1 \times L_3$. Hence, we can choose a and b such that for $i = 2, 4$, either $|L_i| \geq 3$ or $L_i \neq \{a, b\}$. Hence, we can color v_1 and v_3 with a and b respectively and extend this to an L-coloring of G. If $L_1 = L_3$, both L_2 and L_4 are distinct from L_1 and L_3, and G can also be L-colored. □

A vertex of degree k is called a *k-vertex*. A k-wheel is a graph with $k + 1$ vertices $v, v_1, \dots, v_k$ such that vertices $v_1, \dots, v_k$ induce a cycle and they are all adjacent to v. Vertex v is called the center of the k-wheel. Let $N(x)$ denote the neighborhood of a vertex x.

Lemma 2.3. *Let G be a 4-wheel with center v and L be a list assignment on $V(G)$ such that $|L(v)| \geq 4$, every neighbor w of v satisfies $|L(w)| \geq 2$, and there is a neighbor u of v with $|L(u)| \geq 3$. Then G is L-colorable.*

Proof. Let v have neighbors $v_1, \dots, v_4$ with edges v_1v_2, v_2v_3, v_3v_4, v_4v_1, and let $u = v_1$. For each $i \in \{1, \dots, 4\}$, put $L_i = L(v_i)$.

First suppose that $L_2 \cap L_4 \neq \emptyset$. Assign v_2 and v_4 one color from $L_2 \cap L_4$, then greedily color v_3, v and v_1 (in this order) with colors from their lists. The size of the list of each vertex is always one more than the number of distinct colors in its neighborhood, so this gives an L-coloring of G.

Suppose now that $L_2 \cap L_4 = \emptyset$. Since $|L_2 \cup L_4| = 4$, one of L_2, L_4, say L_2, contains a color i that does not appear in L_1. Color v_2 with i, and then greedily color v_3, v_4, v and v_1 (in this order) with colors from their lists. □

Lemma 2.4. *Let G be a 5-wheel with center v and L be a list assignment on $V(G)$ such that $|L(v)| \geq 4$, every neighbor w of v satisfies $|L(w)| \geq 2$, and there are two adjacent neighbors u, u' of v such that $|L(u)| \geq 3$ and $|L(u')| \geq 3$. Then G is L-colorable.*

Proof. Let v have neighbors $v_1, \dots, v_5$ with edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, and let $u = v_1$ and $u' = v_5$. For each $i \in \{1, \dots, 5\}$ put $L_i = L(v_i)$.

Assume first that there is a color $a \in L_4 \setminus L_5$. Color v_4 with a, and then greedily color v_3, v_2, v, v_1 and v_5 (in this order) with colors from their lists. This yields a proper coloring of G. Hence, we can assume in the following that $L_4 \subset L_5$ and (by symmetry) $L_2 \subset L_1$.

Suppose that $L_2 \cap L_4 \neq \emptyset$. Pick any $a \in L_2 \cap L_4$. By the preceding remark we have $a \in L_5 \cap L_1$. Take $b \in L_2 \setminus \{a\}$ and $c \in L_3 \setminus \{a\}$. If $b \neq c$, color vertices v_1, v_4 with a, vertices v_2 and v_3 with b and c respectively, and then greedily color v and v_5 (in this order) with colors from their lists. If $b = c$, we may assume that $L_2 = L_3 = \{a, b\}$. By symmetry, we also have $L_4 = \{a, b\}$ and the previous paragraph implies that L_1 and L_5 both contain a and b. In this case, color vertices v_1, v_3 with a, vertices v_2, v_4 with b, and greedily extend this coloring to v_5 and v (in this order).

We may now assume that $L_2 \cap L_4 = \emptyset$. Without loss of generality, $L_4 = \{a, b\} \subset L_5$ and $L_2 = \{c, d\} \subset L_1$. By symmetry, we may assume that L_3 contains a color distinct from a, b and c. Color v_3 with this color and v_2 with c. Remove c from $L(v)$ and L_1, and remove the color of v_3 from $L(v)$. What remains to color is the diamond induced by v_1, v_4, v_5, v with lists of size 2, except for v_5 which has a list of size 3. By Lemma 2.2, the precoloring of G can be extended to the diamond, and thus G is L-colorable. □

3 $\{$claw$, K_4\}$-free graphs

Recall that the main result of [2] implies that every connected claw-free graph with a stable set of size 3 and no K_4 is 6-colorable. Our main result is the following.

Theorem 3.1. *Let G be a $\{claw, K_4\}$-free graph. Then G is 4-choosable.*

Proof. We prove the theorem by induction on the number of vertices of G. For each vertex v, let $L(v)$ be a list of 4 colors allowed for v. We may assume that G is connected, for otherwise we can handle each component of G separately. If a vertex x has degree at most 3, then, by the induction hypothesis, $G \setminus x$ has an L-coloring, and this can be extended to x since some color in $L(x)$ is not used in $N(x)$. Now let us assume that every vertex has degree at least 4. Suppose that all vertices of G have degree at most

4. If G is not L-colorable, then, by Theorem 2.1, every block of G is either a clique (of size at most 3, since G is K_4-free) or an odd cycle; but then, considering a terminal block, we see that G has a vertex of degree at most 2, a contradiction. So G is L-colorable. Now assume that G has a vertex of degree at least 5. It follows from a classical result in Ramsey theory that every vertex has degree at most 5 (for if a vertex v has degree at least 6, then its neighborhood contains either a clique of size 3 or a stable set of size 3, and then adding v we find a K_4 or a claw). Moreover, if a vertex has degree 5, then its neighborhood induces a 5-cycle; and, by the same argument, if the neighborhood of a vertex contains a 4-cycle, then this vertex does not have a fifth neighbor. We claim that:

> Either G has a 5-vertex whose neighborhood contains two adjacent 4-vertices, or G is one of the graphs G_9, G_{10} or G_{12} depicted in Figure 1. (3)

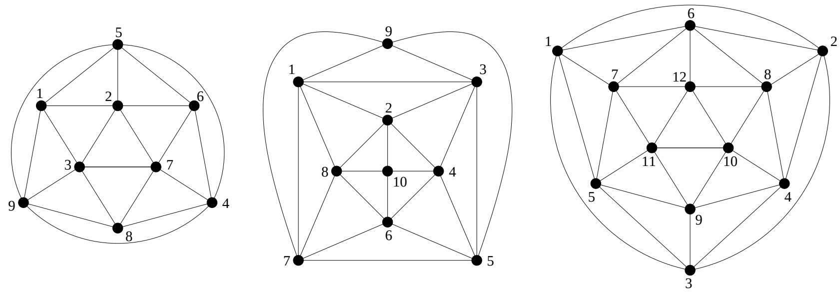

Figure 1: The graphs G_9 (left), G_{10} (center), and G_{12} (right).

Proof. We first assume that all vertices in G have degree 5. Let v_1 be a vertex, and let v_2, v_3, v_5, v_7, v_6 be the 5-cycle induced by its neighborhood. Vertex v_2 has two additional neighbors v_4, v_8 not in $\{v_1, v_2, v_3, v_5, v_6, v_7\}$, and since $N(v_2)$ induces a 5-cycle we may assume by symmetry that this 5-cycle is v_1, v_3, v_4, v_8, v_6. Vertex v_3 has a fifth neighbor distinct from all v_i's, $i \leq 8$, say v_9, and since its neighborhood is a 5-cycle, v_9 is adjacent to v_4 and v_5. Similarly, v_6 has a neighbor v_{12} (distinct from all vertices v_i with $i \leq 9$) that is adjacent to v_7 and v_8. Now v_4 has a neighbor v_{10} (distinct from all vertices v_i with $i \leq 9$ or $i = 12$) that is adjacent to v_8 and v_9. Since the neighborhood of v_8 is a 5-cycle, the vertex v_{10} is also adjacent to v_{12}. At this point the neighborhood of each vertex v_i with $i = 5, 7, 9, 10, 12$ induces a path on four vertices. It follows that these five vertices have a common neighbor, say v_{11}, and G is the icosahedron G_{12}.

Now suppose that G has 5-vertices and 4-vertices. So some 5-vertex v is adjacent to some 4-vertex. Let $v_1, \dots, v_5$ be the neighbors of v, with edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1.

First suppose that v_1 is the only 4-vertex adjacent to v. Since v_2 is a 5-vertex, it has two neighbors u, w not in $\{v, v_1, v_3\}$, and $N(v_2)$ induces a 5-cycle, so we may assume that uv_1, uw, wv_3 are edges (and wv_1, uv_3 are not edges). Likewise, v_3 has a neighbor x not in $\{v, v_2, v_4, w\}$, with edges wx, xv_4 (and xv_2 and wv_4 are not edges). Note that the adjacency relations imply $u \neq x$. Likewise v_4 has a neighbor y not in $\{v, v_3, v_5, x\}$, with edges xy, yv_5, and non-edges yv_3, xv_5, and $w \neq y$. Since v_1 is a 4-vertex, we have $N(v_1) = \{v, v_2, v_5, u\}$. Consequently, since v_5 is a 5-vertex, and $N(v_5)$ must induce a 5-cycle, it must be that $N(v_5) = \{v_1, v, v_4, y, u\}$ and there are edges uv_5, uy. Now u is a 5-vertex, so we must have edge wy, and w and y are 5-vertices. Since $N(x)$ contains a 4-cycle, the vertex x cannot have a fifth neighbor. It follows that, since G is connected, there is no other vertex in G. So G is the graph G_{10}.

Now suppose that v has two neighbors that are 4-vertices and are not adjacent. Say v_2 and v_4 are 4-vertices, while v_1, v_3, v_5 are 5-vertices. By the same argument as above, v_3 has two neighbors u, w not in $\{v, v_2, v_4\}$, with edges uv_2, uw, wv_4 (and uv_4 and wv_2 are not edges). Since v_1 is a 5-vertex, v_2 must have two neighbors in $N(v_1)$, so there is a vertex x such that $N(v_1) = \{v, v_2, u, x, v_5\}$, where ux, xv_5 are edges and uv_5, xv_2 are not edges. Likewise, considering $N(v_5)$, it follows that wv_5, wx are edges. Now $N(x)$ contains a 4-cycle, so x cannot have a fifth neighbor, so G has no other vertex and is a G_9. Thus (3) holds.

By (3), we can break the proof into the following four cases.

Case 1: G has a 5-vertex v whose neighborhood contains two adjacent 4-vertices. Let $v_1, \dots, v_5$ be the neighbors of v, with edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1. Let v_1 and v_5 be 4-vertices. By the induction hypothesis, the graph $G \setminus \{v, v_1, \dots, v_5\}$ is 4-choosable. Pick an L-coloring γ of this graph. For each $i \in \{1, \dots, 5\}$, let N_i be the set of neighbors of v_i in $V(G) \setminus \{v, v_1, \dots, v_5\}$, and let L'_i be the reduced list of v_i, i.e., $L'_i = L(v_i) \setminus \gamma(N_i)$. Thus we have $|L'_i| \geq 2$ for each $i \in \{1, \dots, 5\}$ and $|L'_j| \geq 3$ for $j \in \{1, 5\}$ because $|N_j| = 1$. Note that $\{v, v_1, \dots, v_5\}$ induces a 5-wheel. So this subgraph and the reduced lists satisfy the hypothesis of Lemma 2.4, and consequently we can extend γ to an L-coloring of G using for each vertex v_i a color from L'_i and for v a color from $L(v)$.

Case 2: G is G_9. For each $i \in \{1, \ldots, 9\}$, let v_i be the vertex labeled i on Figure 1 (left), and let $L_i = L(v_i)$. If all the lists are equal, then since G_9 is 4-colorable, it is also L-colorable. So we can assume that there are i, j such that $L_i \neq L_j$. Moreover, since the set of edges between vertices of degree 4 and vertices of degree 5 forms a spanning subgraph of G, we may assume that v_i and v_j have degree 4 and 5, respectively; and since all edges between vertices of degree 4 and 5 play the same role we may assume that $L_1 \neq L_5$. We assign a color $a \in L_1 \setminus L_5$ to v_1 and then greedily color v_2 and v_3 with colors from their lists. For each $i \in \{4, \ldots, 9\}$, let L'_i be the list obtained from L_i by removing the colors of the neighbors of v_i in $\{v_1, v_2, v_3\}$. Observe that $|L'_7| \geq 2$, $|L'_9| \geq 2$, $|L'_6| \geq 3$, $|L'_8| \geq 3$ and $|L'_4| \geq 4$. By the choice of color a, we have $|L'_5| \geq 3$. The graph induced by $\{v_i \mid 4 \leq i \leq 9\}$ is a 5-wheel centered at v_4, so Lemma 2.4 implies that the precoloring of $\{v_1, v_2, v_3\}$ can be extended to an L-coloring of G.

Case 3: G is G_{10}. For each $i \in \{1, \ldots, 10\}$, let v_i be the vertex labeled i on Figure 1 (center), and let $L_i = L(v_i)$. First suppose that $L_1 \neq L_2$. We color v_1 with any $a \in L_1 \setminus L_2$ and then greedily color v_9, v_3, v_5 and v_7 (in this order) with colors from their lists. For each even i, let L'_i be the list obtained from L_i by removing the colors of the neighbors of v_i in the set $V_o = \{v_j,\ j \text{ odd}\}$. Observe that $|L'_i| \geq 2$ for each $i \in \{4, 6, 8\}$, $|L'_{10}| \geq 4$, and $|L'_2| \geq 3$ since $a \notin L_2$. The graph induced by the set $\{v_j,\ j \text{ even}\}$ is a 4-wheel centered at v_{10}, so Lemma 2.3 implies that the precoloring of V_o can be extended to an L-coloring of G_{10}.

Now we may assume that $L_1 = L_2$ and, similarly $L_i = L_{i+1}$ for all $i \leq 7$. It follows that the eight sets $L_1, \ldots, L_8$ are equal. Let $L_1 = \cdots = L_8 = \{0, 1, 2, 3\}$. For every $i \leq 8$, assign color $i \pmod 4$ to v_i, and then greedily color v_9 and v_{10} with colors from their lists (each of them has only two distinct colors in its neighborhood). This yields an L-coloring of G.

Case 4: G is G_{12}. For each $i \in \{1, \ldots, 12\}$, let v_i be the vertex labeled i on Figure 1 (right), and let $L_i = L(v_i)$. Note that every edge of G is in exactly two triangles and that G is vertex-, edge-, and triangle-*transitive*, i.e., for any two vertices (resp. edges, triangles) α and β, there is an automorphism of

G that maps α to β. Moreover, G is planar and thus 4-colorable. We claim that we may assume that:

$$\text{If } \{v_i, v_j, v_k\} \text{ is any triangle in } G\text{, then } L_i \subseteq L_j \cup L_k. \tag{4}$$

Proof. By transitivity, we may assume that $i = 1$ and $\{j, k\} = \{2, 3\}$. Suppose that there is a color a in $L_1 \setminus (L_2 \cup L_3)$. We assign a to v_1 and then greedily color $v_5, v_6, v_7, v_{11}, v_{12}$ (in this order) with colors from their lists. For each uncolored vertex v_i ($i \in \{2, 3, 4, 8, 9, 10\}$), let L'_i be the list obtained from L_i by removing the colors of the neighbors of v_i in $\{v_1, v_5, v_6, v_7, v_{11}, v_{12}\}$. The graph induced by the uncolored vertices is a 5-wheel centered at v_4. Observe that $|L'_i| \geq 2$ for each $i \in \{8, 9, 10\}$, $|L'_4| \geq 4$, and $|L'_2| \geq 3$ and $|L'_3| \geq 3$ because $a \notin L_2 \cup L_3$. By Lemma 2.4, the precoloring can be extended to an L-coloring of G_{12}. Therefore we may assume that (4) holds.

Let the *weight* of an edge v_iv_j of G be the cardinality of $L_i \cap L_j$. Hence the weight of any edge is at most 4. Suppose that there is an edge v_iv_j of weight 4, i.e., $L_i = L_j$. If all edges have weight 4, then all lists are equal, and since G_{12} is 4-colorable, it is also L-colorable. So suppose that some other edge has weight at most 3. Then there exists a triangle $v_iv_jv_k$ such that v_iv_j has weight 4 and v_iv_k has weight at most 3, so there is a color $a \in L_k \setminus (L_i \cup L_j)$; but then $\{v_k, v_i, v_j\}$ violates (4).

We may now assume that every edge has weight at most 3. Suppose that there is an edge, say v_1v_2, of weight at most 2. If the weight of v_1v_2 is 0 or 1, then, by (4), L_3 contains the symmetric difference of L_1 and L_2, which has size at least 6, a contradiction. If the weight of v_1v_2 is 2, let $L_1 = \{1, 2, 3, 4\}$ and $L_2 = \{1, 2, 5, 6\}$. By (4), we must have $L_3 = L_6 = \{3, 4, 5, 6\}$, and then, by (4) again, we must have $L_5 = L_7 = \{1, 2, 5, 6\}$, a contradiction.

Therefore all edges have weight 3. We may assume that $L_1 = \{1, 2, 3, 4\}$ and $L_2 = \{1, 2, 3, 5\}$. Then:

$$\text{Every vertex } v_i \text{ of } G \text{ satisfies } L_i \subset \{1, 2, 3, 4, 5\}. \tag{5}$$

We prove this by induction on i. For $i = 2$, this is our assumption. If $i \geq 3$, it is easy to see that there exists a triangle $\{v_i, v_j, v_k\}$ with $k < j < i$. If L_i contains some color $a \notin \{1, 2, 3, 4, 5\}$, then by (4) at least one of L_j, L_k contains a, a contradiction. Thus (5) holds.

By (5), the intersection of any three lists has size at least 2. Hence L_1, L_8 and L_{11} have a color a in common, and L_2, L_9 and L_{12} have a color b in common with $b \neq a$. Color vertices v_1, v_8, v_{11} with a, vertices v_2, v_9, v_{12} with b, and remove a and b from the lists of the remaining vertices. The remaining vertices induce a path and their reduced lists all have size at least 2, so the precoloring can be extended to an L-coloring of G_{12}. This completes the proof of the theorem. □

4 $\{$claw, $K_5\}$-free graphs

In this section, all indices are considered modulo 8. Let W_0 be the graph with eight vertices $w_0, w_1, \ldots, w_7$ and edges w_iw_{i+1} and w_iw_{i+2} for each i. (So W_0 is the complement of the *Wagner graph*). Let W_1 be the graph W_0 plus one edge $w_\ell w_{\ell+4}$ (for some $\ell \in \{0, \ldots, 7\}$), and let W_2 be the graph W_0 plus two edges $w_\ell w_{\ell+4}$ and $w_{\ell+1}w_{\ell+5}$ (for some $\ell \in \{0, \ldots, 7\}$). These graphs are depicted in Figure 2. Greenwood and Gleason [5] and Kéry [7] proved that $R(4, 3) = 9$ and determined the corresponding Ramsey graphs. More precisely:

Theorem 4.1 ([5, 7])**.** *Every graph with* 9 *vertices contains a clique of size* 4 *or a stable set of size* 3*. Moreover if a graph on* 8 *vertices contains no clique of size* 4 *and no stable set of size* 3*, then it is isomorphic to* W_0*,* W_1 *or* W_2*.*

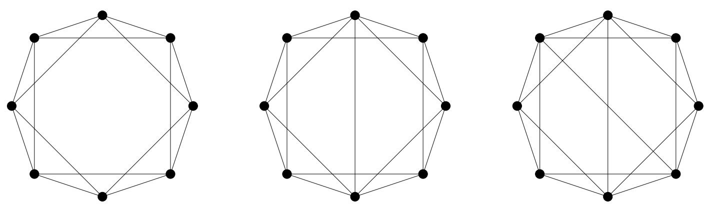

Figure 2: Graphs W_0 (left), W_1 (center), and W_2 (right).

Now we can prove:

Theorem 4.2. *Let* G *be a* $\{$*claw*$, K_5\}$*-free graph. Then* G *is* 7*-choosable.*

Proof. We prove the theorem by induction on the number of vertices of G. For each vertex v of G, let $L(v)$ be a list of 7 colors allowed for v. We may assume that G is connected, for otherwise we can handle each component of G separately.

If G contains a vertex v of degree at least 9 then by Theorem 4.1, the neighborhood of v contains a stable set of size 3 (and thus G contains a claw) or a clique of size 4 (in which case G has a clique of size 5). In both cases we obtain a contradiction, since G is claw-free and has no clique of size 5. Hence, every vertex of G has degree at most 8.

If some vertex x has degree at most 6, then, by the induction hypothesis, $G \setminus x$ has an L-coloring, and this can be extended to x since some color in $L(x)$ is not used in $N(x)$. Now let us assume that every vertex has degree at least 7. Suppose that all vertices of G have degree at most 7. If G is not L-colorable, then, by Theorem 2.1, every block of G is a clique (of size at most 4, since G is K_5-free) or an odd cycle; but then, considering a terminal block, we see that G has a vertex of degree at most 3, a contradiction; so G is L-colorable.

Now assume that G has a vertex x of degree 8. Again, since G is claw-free and has no clique of size 5, the graph induced by the neighborhood of x has no stable set of size 3 and no clique of size 4. It follows by Theorem 4.1 that the neighborhood of x induces one of W_0, W_1 or W_2. We call W the neighborhood of x, with vertices $w_0, \dots, w_7$ and edges as at the beginning of this section (where, for some fixed $\ell \in \{0, \dots, 7\}$, $w_\ell w_{\ell+4}$ may be an edge and if it is an edge then $w_{\ell+1} w_{\ell+5}$ may also be an edge).

Let $S = \{x, w_0, \dots, w_7\}$. By the induction hypothesis, the graph $G \setminus S$ is 7-choosable. Consider an L-coloring of this graph, and for each $i \in \{0, \dots, 7\}$ let L_i be the list obtained from $L(w_i)$ by removing the colors of the neighbors of w_i (outside S). Since every vertex has degree at most 8, we have $|L_i| \geq 4$ for each $i \in \{0, \dots, 7\}$. Moreover, if $w_j w_{j+4}$ is an edge, then $|L_j| \geq 5$. In order to prove the theorem, it is now sufficient to properly color each w_i with a color from L_i and x with a color from $L(x)$. We call *good pair* any pair $\{w_i, w_{i+3}\}$ such that $L_i \cap L_{i+3} \neq \emptyset$.

First suppose that there is no good pair. If the lists of the vertices of S have a set of distinct representatives, then we are done (assign to each vertex the color that is the distinct representative of its list). In the opposite case, by Hall's theorem, there exists a set $Z \subseteq S$ such that the union U of all lists of vertices $z \in Z$ satisfies $|U| < |Z|$. Since each such list has size at least 4, we have $|U| \geq 4$, so $|Z| \geq 5$. If Z contains x, then $|U| \geq 7$, so $|Z| \geq 8$. In any case Z must contain at least five vertices from $S \setminus x$. This implies that Z contains two vertices w_i, w_{i+3} for some $i \in \{0, \dots, 7\}$. Since there is no good pair, L_i and L_{i+3} are disjoint, so $|U| \geq 8$, which means that $|U| = 8$ and $Z = S$. Moreover U is partitioned into two sets U_1 and U_2 of size 4 such that $L_1 = U_1$, $L_4 = U_2$ (because $\{w_1, w_4\}$ is not a good pair), and similarly $L_7 = U_1$, etc. Hence $L_1 = L_3 = L_5 = L_7 = U_1$ and $L_0 = L_2 = L_4 = L_6 = U_2$. In particular, all these lists have size 4, so each pair $w_i w_{i+4}$ is a non-edge (W is isomorphic to W_0). Color vertices w_1, w_5 with some color $a \in U_1$, vertices w_3, w_7 with some color $b \in U_1 \setminus a$, vertices w_2, w_6 with some color $c \in U_2$ and vertices w_4, w_0 with some color $d \in U_2 \setminus c$. Finally, assign to x any color from $L(x) \setminus \{a, b, c, d\}$. This yields an L-coloring of G.

We may now assume that there is a good pair, say $\{w_0, w_5\}$. Let $S' = S \setminus \{w_0, w_5\}$. Pick a color $a \in L_0 \cap L_5$, set $L'_i = L_i \setminus \{a\}$ for each $i \in \{1, 2, 3, 4, 6, 7\}$ and $L'_x = L(x) \setminus \{a\}$. We have $|L'_i| \geq 3$ for each $i \in \{1, 2, 3, 4, 6, 7\}$ and $|L'_x| \geq 6$. Moreover, if $w_i w_{i+4}$ is an edge, then $|L'_i| \geq 4$. Our goal is to color every vertex v of S' with a color from the corresponding set L'_v. We call *nice pair* any pair $\{w_i, w_{i+3}\}$ of vertices of $S' \setminus \{x\}$ such that $L'_i \cap L'_{i+3} \neq \emptyset$.

Suppose that there is no nice pair. If the lists of the vertices of S' have a set of distinct representatives then we are done. Otherwise, by Hall's theorem, there exists a set $Z \subseteq S'$ such that the union U of all lists of vertices $z \in Z$ satisfies $|U| < |Z|$. Since each such list has size at least 3, we have $|U| \geq 3$, so $|Z| \geq 4$. If $x \in Z$, then $|U| \geq 6$ and $|Z| \geq 7$. In any case, Z contains at least four vertices of $S' \setminus x$. This implies that Z contains w_i, w_{i+3} for some $i \in \{1, 3, 7\}$. Since there is no nice pair, L_i and L_{i+3} are disjoint, so $|U| \geq 6$, which means that $|U| = 6$ and $Z = S'$. Moreover, U can be partitioned into two sets U_1 and U_2 of size 3 such that $L'_1 = L'_3 = L'_7 = U_1$, and $L'_2 = L'_4 = L'_6 = U_2$. In particular, all these lists have size 3, so the pairs $w_2 w_6$ and $w_3 w_7$ are non-edges. Color w_2, w_6 with a color $b \in U_2$ and w_3, w_7 with a color $c \in U_1$. This coloring can be extended greedily to w_1, w_4, and x, which yields an L-coloring of G.

We may now assume that there is a nice pair $P = \{w_i, w_{i+3}\}$. Up to symmetry, P is either $\{v_2, v_7\}$, $\{v_1, v_4\}$ or $\{v_1, v_6\}$. In any case, color w_i and w_{i+3} with a color $b \in L'_i \cap L'_{i+3}$, and remove b from the lists of the neighbors of w_i and w_{i+3}. Let $S'' = S' \setminus P$. If the four vertices of S'' can be colored with colors from their reduced lists (which have size at least 2), then so does x (because its reduced list has size at least 5). If $P = \{v_1, v_4\}$ or $\{v_2, v_7\}$, then the graph induced by S'' is a subgraph of the 4-cycle, so it is 2-choosable. In the remaining case, $P = \{v_1, v_6\}$, so $S'' = \{w_2, w_3, w_4, w_7\}$, which induces a triangle plus vertex w_7 that has a most one neighbor in the triangle. This subgraph is not 2-choosable if and only if w_2, w_3, w_4 have the same reduced list of size 2. This occurs only if both a and b were removed from their original lists, thus we have $b \in L(w_4)$, which implies that $\{w_1, w_4\}$ was also a nice pair, a case that has already been settled. Hence, G is L-colorable. This completes the proof of the theorem. □

Conclusion

The proofs of Theorems 3.1 and 4.2 are based on the fact that for small values of k, Ramsey graphs corresponding to $R(k, 3)$ are known and the number of such graphs is small. It is unlikely that our techniques will be usable for larger values of k. Still it would be interesting to establish that $\chi(\mathcal{C}_6) = ch(\mathcal{C}_6)$ and to know the exact value of this number. As Table 1 shows, we only know that $9 \leq \chi(\mathcal{C}_6) \leq 10$. Since $R(5, 3) = 14$, it follows that every graph of $\mathcal{C}_6$ has maximum degree at most 13, and we can easily derive from Theorem 2.1 that $ch(\mathcal{C}_6) \leq 13$.

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