The Existential Theory of the Reals

Arnaud de Mesmay (CNRS, LIGM, Université Gustave Eiffel, Paris)

Graphs and complexity day, ENS Lyon, April 7, 2025

Why should I care?

The *Existential Theory of the Reals* is

1. The problem of finding whether a set of polynomial (in)equations has a solution over the reals.
2. The complexity of deciding whether a pseudo-line arrangement can be stretched.
3. The computational counterpart of the Mnëv universality theorem.

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The actual “definition”

The Existential Theory of the Reals ($\exists\mathbb{R}$) is the complexity class that appears naturally for nice discrete problems that

- look like they should be in **NP**
- but there are annoying precision or algebraic issues to bound the size of the certificate.

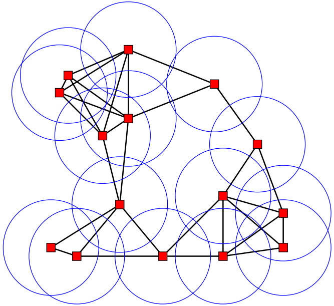
Example 1: Unit disk graphs

Unit Disk Graph Recognition

Input: A graph G.

Output: Is G the intersection graph of unit disks in the plane?

How can you certify that a graph is a unit disk graph?



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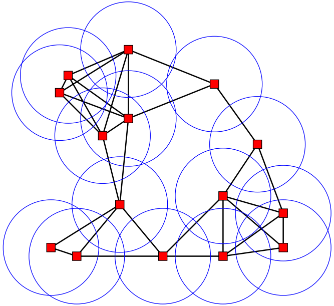

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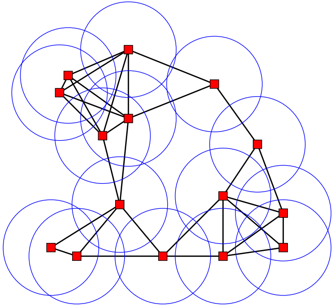

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Theorem (Kang-Muller '11)

UNIT DISK GRAPH RECOGNITION *is* $\exists\mathbb{R}$*-complete.*

Example 2: Matrix Rank problems

MINIMUM MATRIX RANK

Input: A matrix with $0, 1$ entries and some unknowns $(x_1, \ldots, x_n)$ and an integer k.

Output: Can one find real values for $(x_1, \ldots, x_n)$ so that the matrix has rank at most k?

How to certify low-rank completion of matrices?

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How to certify low-rank completion of matrices?

- Easy! You can "just" give the values of the missing entries.

Theorem (Buss-Frandsen-Shallit '99)

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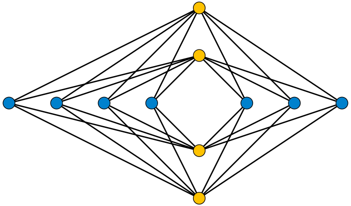
Example 3: Crossing Numbers

Rectilinear Crossing Number

Input: A graph G and an integer k.

Output: Is there a straight-line drawing of G with at most k crossings?



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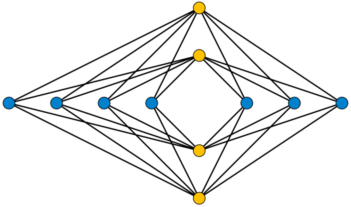

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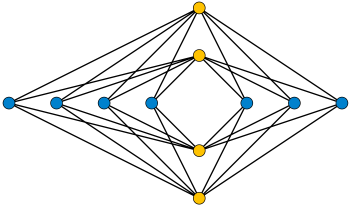

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Corollary

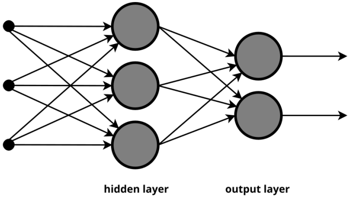
The crossing number and the rectilinear crossing number of a graph can be different *(if $\exists\mathbb{R} \neq$ **NP**).*

Example 4: The AI slide

Training Neural Networks

Input: A set of data points, a neural network architecture, a cost function, a threshold.

Output: Are there biases and weights such that the total error is below the threshold?



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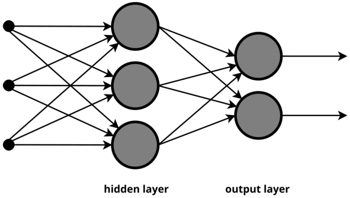

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Theorem (Abrahamsen-Kleist-Miltzow '21, BHJMW '22)

Training Neural Networks *is* $\exists\mathbb{R}$*-complete.*

Back to the first slide

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Plan of the talk

The *plan of the talk* is

1. Algebraic aspects (a.k.a. how to prove $\exists\mathbb{R}$-membership).
2. Stretchability (a.k.a. how to prove $\exists\mathbb{R}$-hardness).
3. Some more things to know about $\exists\mathbb{R}$.

1. Algebraic aspects

Existential Theory of the Reals (ETR)

Input: A system of polynomial (in)equations.
Output: Is there a solution over the reals?

The complexity of the equations (degree, size of the coefficients, number of summands) is counted in the input.

$$\begin{aligned} x^4 + y^2 + 24 \geq &\quad 0 \\ x^3 + z = &\quad 22 \\ x^4 + x^7 + x^5 = &\quad 5 \end{aligned}$$

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The Existential Theory of the Reals as a complexity class

The Existential Theory of the Reals ($\exists\mathbb{R}$) is the complexity class made of all problems that can be (many-to-one) reduced to the Existential Theory of the Reals (ETR).

Intuitively, $\exists\mathbb{R}$ is the set of problems that can be encoded as the real solutions of polynomial inequations.

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Corollary

The Existential Theory of the Reals is complete for the Existential Theory of the Reals.

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Corollary

ETR *is* $\exists\mathbb{R}$*-complete.*

Where is $\exists\mathbb{R}$? (1)

Theorem

$$\mathbf{NP} \subseteq \exists\mathbb{R}.$$

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Proof:

- We encode 3-SAT variables as 0 or 1.
- We enforce these values with equations $x(x-1)=0$.
- We encode variable negation with $y = 1 - x$.
- We encode 3-SAT clauses $x \vee y \vee z$ as $x + y + z \geq 1$

That's it.

Where is $\exists\mathbb{R}$? (2)

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There is no solution in radicals (square roots, cubic roots, etc.) to general polynomial equations of degree five or higher.

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- But ETR is not about *finding* solutions, just about *deciding* if there is one.
- For example, $ax^2 + bx + c = 0$ with $a \neq 0$ has a real solution if and only if $b^2 - 4ac \geq 0$.
- Note that this involves no radicals.
- We have reduced the *existential* equation $\exists x, ax^2 + bx + c = 0$ to the *quantifier-free* equation $b^2 - 4ac \geq 0$.

Quantifier elimination

The simple degree-two example generalizes completely:

Theorem (Tarski-Seideberg quantifier elimination)

Any formula construction with polynomial equations, inequations, logical connections $\vee$, $\wedge$, $\neg$ *and quantifiers* $\exists$ *and* $\forall$ *is equivalent to a similar formula without quantifiers.*

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- When there are only *existential quantifiers*, which is the case for ETR, the complexity improves to **PSPACE** [Canny '88].

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Be careful

Quantifier elimination may turn equalities into inequalities.

So, where is $\exists\mathbb{R}$?

Theorem

$$\mathbf{NP} \subseteq \exists\mathbb{R} \subseteq \mathbf{PSPACE}.$$

It is widely conjectured that the two inclusions are strict, but nothing is known about that.

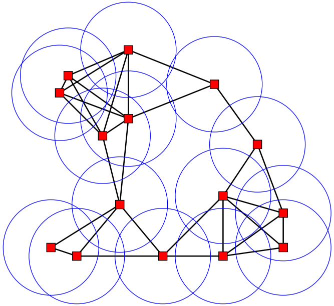
How to prove $\exists\mathbb{R}$ membership?

By definition, to prove $\exists\mathbb{R}$membership, it suffices to encode the problem as a set of algebraic (in)equations.

Example: Unit Disk Graph Recognition

Given a graph $G = (V, E)$:

- For each vertex v_i, use a pair of unknowns (x_i, y_i).
- For each edge $e = (v_i, v_j)$, use an equation $(x_i - x_j)^2 + (y_i - y_j)^2 \leq 1$.
- For each non-edge $\neg e = (v_i, v_j)$, use an equation $(x_i - x_j)^2 + (y_i - y_j)^2 > 1$.



Theorem

Unit Disk Graph Recognition $\in \exists\mathbb{R}$.

How to prove $\exists\mathbb{R}$ membership faster?

- When proving NP-membership, we generally do not write a SAT-formula encoding the problem. Instead, we prove that it is easy to certify a solution, i.e., that it can recognized by a non-deterministic Turing machine in polynomial time.

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- Likewise:

Theorem (Erickson, van der Hoog, Miltzow '20)

$\exists\mathbb{R}$ *membership is equivalent to being able to verify a solution in polynomial time on a Real RAM machine.*

Intuitively, a real RAM machine is like a word RAM model (or a Turing machine) that is allowed to

- manipulate real numbers, and
- operations on them (addition, multiplication, substraction, division, square roots)

in constant time.

It is unwise to allow downcasting reals to integers (e.g., allowing the floor function $\lfloor x \rfloor$).

2. Proving $\exists\mathbb{R}$-hardness

- As for every complexity class, one proves $\exists\mathbb{R}$-hardness of a problem P by reducing an $\exists\mathbb{R}$-complete problem to it.
- Recall that ETR is $\exists\mathbb{R}$-complete. This is not impractical and for example can be used to prove $\exists\mathbb{R}$-hardness of
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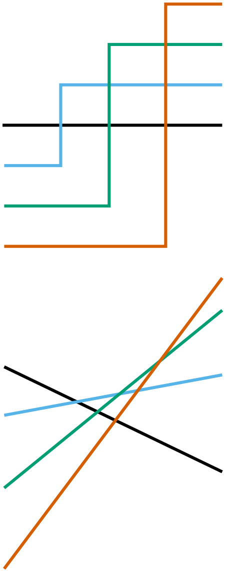
- But for most problems, in particular for those of a *geometric* nature, it is more common to reduce from another problem: Stretchability.

Stretchability of pseudoline arrangements

A *pseudoline arrangement* is a collection of x-monotone curves in the plane that each cross pairwise exactly once.

STRETCHABILITY

Input: A pseudoline arrangement (for example described with polygonal lines).
Output: Is it *homeomorphic* to an arrangement of straight lines?



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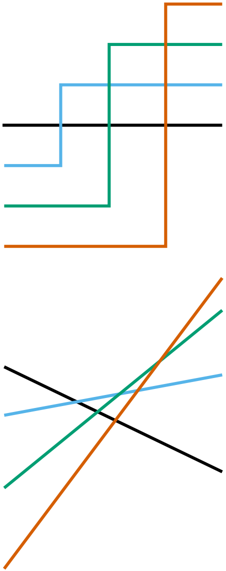

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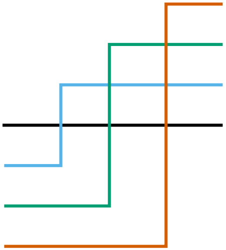

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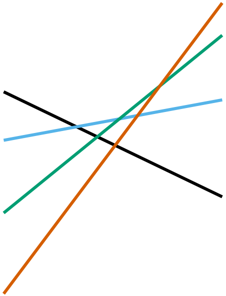
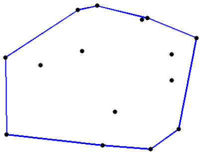


Parenthesis on abstract data structures in computational geometry

- *Computational geometry* investigates algorithms with geometric input.

Example: Given a set of points in $\mathbb{R}^2$, compute their convex hull.



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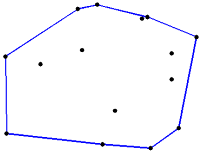

- Most of these problems do not actually rely on the coordinates of the points, but only on *orientation predicates*: is A to the left, on or to the right of the line (BC)?
- So it makes sense to use these predicates as the data structure to work with. This is called a *chirotope*, or *abstract order type*, or *oriented matroid of rank* 3.

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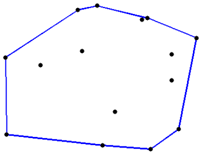

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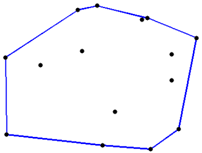

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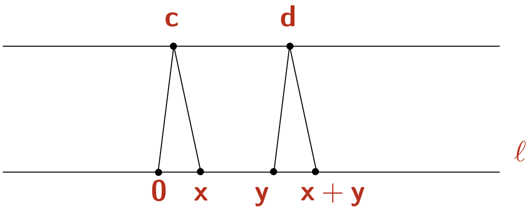
- Then the question arises: given a chirotope, does it correspond to an actual set of points?
- Via the duality $(a, b) \mapsto ax - b$ mapping points to lines, this is equivalent to the STRETCHABILITY problem.

$\exists\mathbb{R}$-completeness of STRETCHABILITY

Theorem (Mnëv '88, Shor '91)

STRETCHABILITY *is* $\exists\mathbb{R}$*-complete.*

- The key insight comes from the *von Staudt constructions*, allowing to encode algebraic operations in line arrangements.



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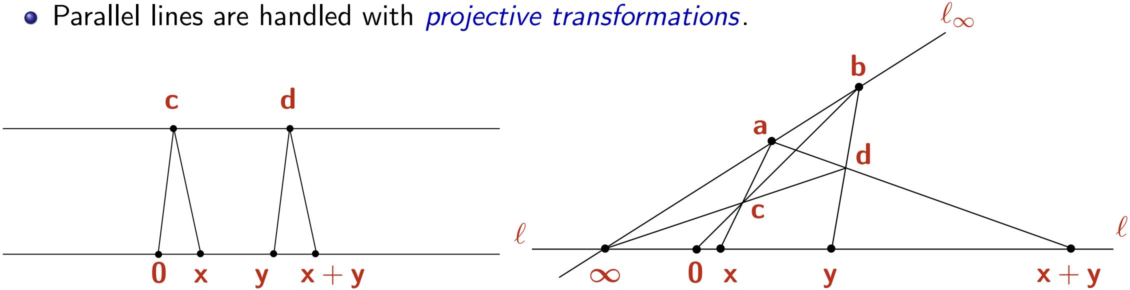

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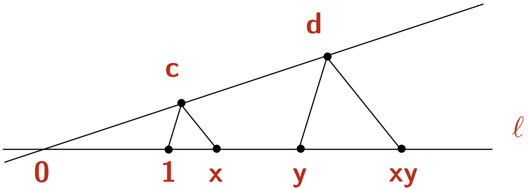

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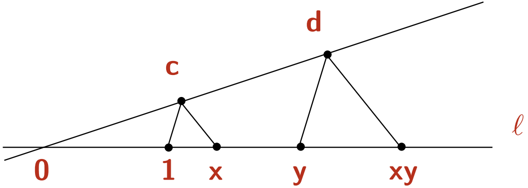

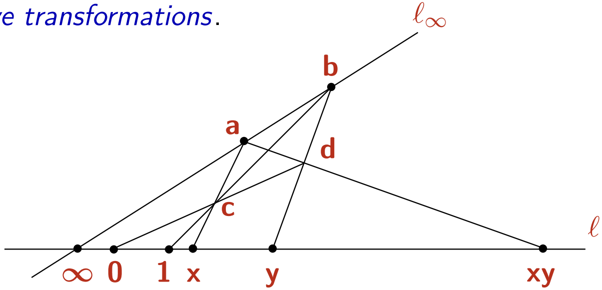

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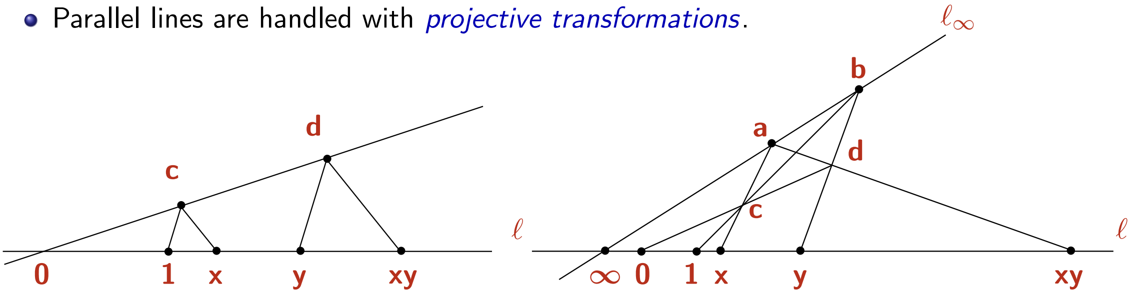
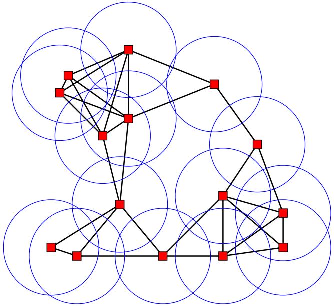


Using these gadgets, the solvability of an ETR formula translates into the stretchability of some line arrangement, but there are **many** tricky details.

3. A few more things to know about $\exists\mathbb{R}$: Precision issues

How precise do you need to be if you actually want to describe the coordinates of a unit disk graph?



$\exists\mathbb{R}$-complete problems generally require *doubly exponential precision*.

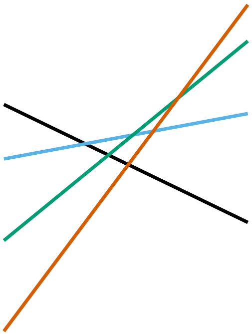
- This comes from the fact that one can encode numbers of size 2^{2^n} or $2^{2^{-n}}$ with only n equations.
- Conversely, real algebraic geometric theorems shows that for open sets, doubly exponential precision is generally enough.
- Similarly, for closed sets, irrational numbers are generally required.

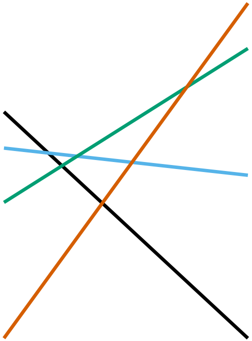
3. A few more things to know about $\exists\mathbb{R}$: Universality

Conjecture (Ringel '56)

If two line arrangements have the same chirotope, then they are isotopic, i.e., one can be deformed into the other.

Seems intuitive...





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Theorem (Mnëv '86)

Any semi-algebraic variety is stably equivalent to the realization space of some pseudoline arrangement.

- A semi-algebraic variety is just a set of solutions of an ETR formula.
- Stable equivalence would bring us beyond the scope of this tutorial, but ...
- ... in a nutshell, universality means that the solution space of $\exists\mathbb{R}$-complete problems is generally *horribly complicated* from an algebraic, geometric and topological point of view.
- In particular, it is in general **not** connected, nor simply connected, nor contractible or anything like that.

3. A few more things to know about $\exists\mathbb{R}$: Problems that are not $\exists\mathbb{R}$-complete

The actual "definition"

The Existential Theory of the Reals ($\exists\mathbb{R}$) is the complexity class that appears naturally for nice discrete problems that

- look like they should be in **NP**
- but there are annoying precision or algebraic issues to bound the size of the certificate.

Some non-examples:

- Problems that are *overconstrained*: for example, deciding whether a weighted graph represents distances in $\mathbb{R}^d$ is $\exists\mathbb{R}$-complete, but not if the graph is complete.
- Problems easy to optimize. For example, computing the geometric median (point that minimizes sums of distances) of points in $\mathbb{R}^d$ runs into algebraic issues, but it is probably not $\exists\mathbb{R}$-complete.
- In particular, the famous SUM OF SQUARE ROOTS PROBLEM: given integers $a_1, \ldots, a_n$ and k, is $\sum \sqrt{a_i} \geq k$? is **not** believed to be $\exists\mathbb{R}$-complete.

Some concluding words

- There are a **lot** of very different $\exists\mathbb{R}$-complete problems.
- Still a **lof** of open questions.
- Many topics I have not touched on, among them:
  - $\forall\exists\mathbb{R}$-completeness and higher levels of the hierarchy,
  - connections to Blum-Shub-Smale models,
  - $\exists\mathbb{Z}$ (undecidable), $\exists\mathbb{C}$ (easier), $\exists\mathbb{Q}$ (unknown decidability),
  - actual algorithms to solve systems of polynomial inequations,
  - ...
- To learn more about this topic: read
  - The Existential Theory of the Reals as a Complexity Class: A Compendium, by Schaefer, Cardinal and Miltzow
  - Segment intersections graphs and $\exists\mathbb{R}$, by Matoušek

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Thank you! Any questions?