At the other end of the (directed) path: the complexity classes PPA and PPAD

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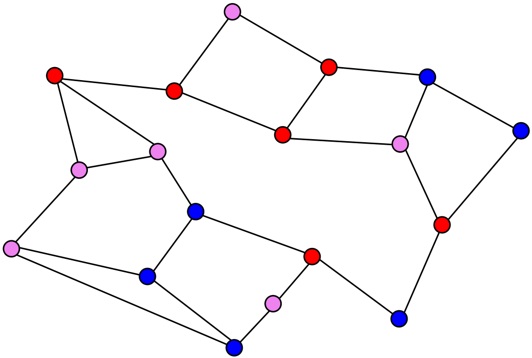
Rainbow independent sets

Theorem Haxell 1995

In a colored graph with maximum degree Δ, there always exists an independent set intersecting every color class of size at least 2Δ.



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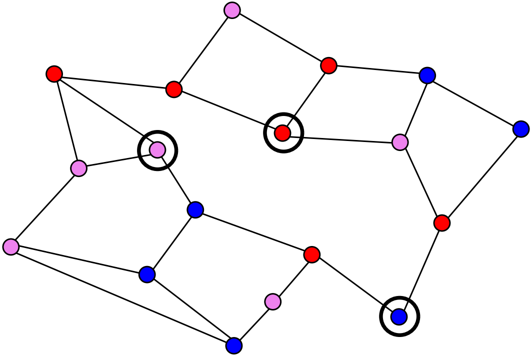


Rainbow independent sets

Theorem Haxell 1995

In a colored graph with maximum degree Δ, there always exists an independent set intersecting every color class of size at least 2Δ.

- No known polynomial algorithm for computing such an independent set
- No known hardness result
- From the NP perspective: polynomial problem, since answer always 'yes'
- Complexity class TFNP (introduced by Meggido and Papadimitriou 1989)

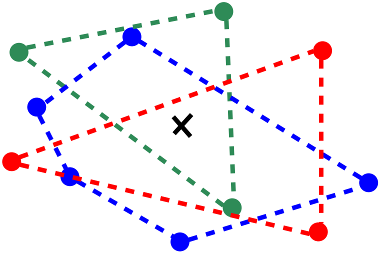
Plan

The complexity class TFNP

TFNP $\simeq$ class of function problems for which a solution is guaranteed to exist

examples:

- colorful independent set as in Haxell's theorem
- prime factorization
- Nash equilibria
- colorful Carathéodory



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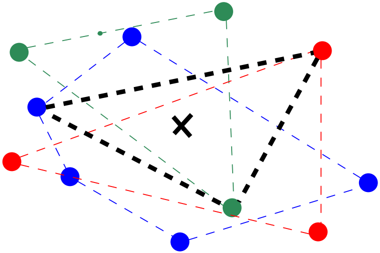


The complexity classes FP, FNP, TFNP

Function problem (aka search problem) R: given I, find s such that $(I, s) \in R$

$R \in$ FNP if

- $(I, s) \in R$ can be decided in polynomial time.
- $(I, s) \in R \Longrightarrow |s| \leqslant f(|I|)$ for some polynomial f.

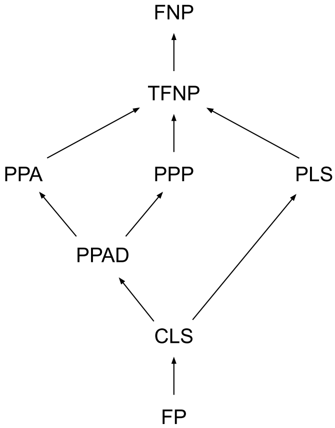
$R \in$ FP if $R \in$ FNP and can be solved in polynomial time.

$R \in$ TFNP if R is total, i.e., for every I, there is an s such that $(I, s) \in R$.

The complexity class TFNP

Theorem Meggido and Papadimitriou 1989

There exists an FNP-complete problem in TFNP if and only if NP = coNP.



No hope to show hardness of a TFNP problem by showing FNP-completeness

Existence of a TFNP-complete problem is unlikely (Meggido and Papadimitriou 1989)

Plan

1. TFNP
2. PPA, PPAD
   - PPA
   - PPAD
3. Completeness
4. PPA-k
5. Open problems
6. Concluding remarks

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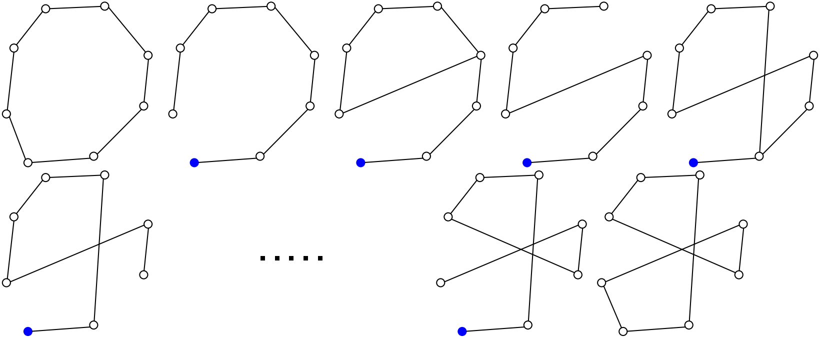
The complexity class PPA

Theorem Smith and Tutte 1946

If a cubic graph has a Hamiltonian cycle, then there is a second Hamiltonian cycle.



The complexity class PPA

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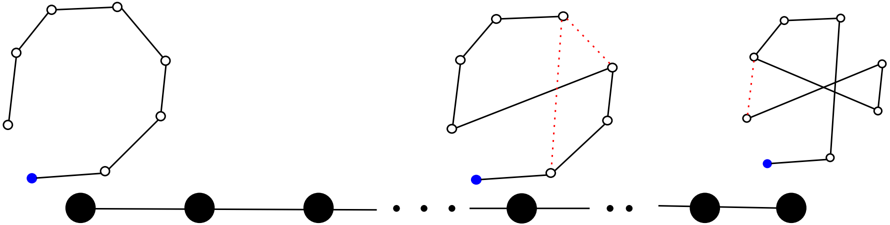


The complexity class PPA, formal definition

Problem `Leaf`.
Input. A graph where every vertex is of degree at most two; a degree-one vertex
Output. Another degree-one vertex

(The graph can be potentially huge; described by a circuit.)

> PPA = class of function problems that polynomially reduce to `Leaf`

Introduced by Papadimitriou (1992)

Polynomial reduction

Let $R, S \in \mathrm{TFNP}$

R polynomially reduces to S if there exist polynomially-computable functions f and g such that

$$\big(f(I), s\big) \in S \implies \big(I, g(I, s)\big) \in R\,.$$

Definition still valid if $R \in \mathrm{FNP}$: polynomial reduction is then an efficient proof that R is total.

Another-Hamiltonian-Cycle

Problem `Another-Hamiltonian-Cycle`.
Input. A cubic graph; a Hamiltonian cycle
Output. Another Hamiltonian cycle

`Another-Hamiltonian-Cycle` $\in$ PPA

The complexity class PPA, alternative formal definition

PPA can be alternatively defined as follows (Papadimitriou 1992).

Problem `Odd-Degree-Vertex`.
Input. A graph; an odd-degree vertex
Output. Another odd-degree vertex

(The graph can be potentially huge; described by a circuit.)

PPA $=$ class of function problems that polynomially reduce to `Odd-Degree-Vertex`

Plan

The complexity class PPAD

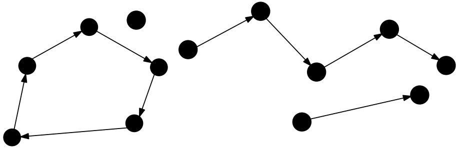
Problem `End-Of-The-Line`.

Input. A directed graph formed by vertex-disjoint directed paths and directed cycles; a vertex u that is the origin of a path

Output. A vertex u' that is the end of a path or the origin of another path

(The graph can be potentially huge; described by a circuit.)

PPAD = class of function problems that can be polynomially reduced to `End-Of-The-Line`



Introduced by Papadimitriou (1992)

Relation between PPA and PPAD

- PPAD $\subseteq$ PPA (immediate)
- Strongly believed that the inclusion is strict

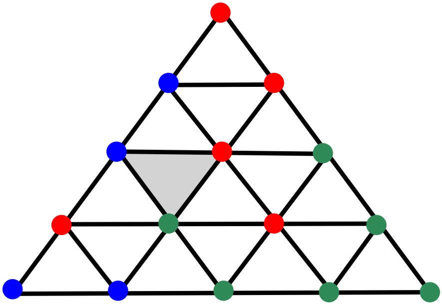
Sperner's lemma

Lemma Sperner 1928

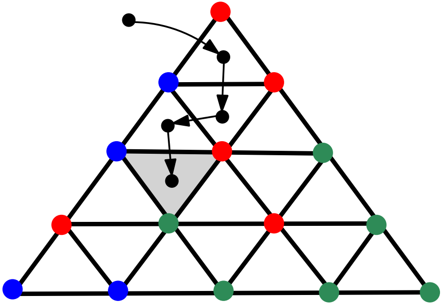
Consider a triangulation of a triangle, whose vertices are colored with blue, red, green. Suppose that

- ⋆ the extreme points of the triangle get distinct colors.
- ⋆ every vertex in a side has the color of one of the endpoints.

Then there exists a small *colorful* triangle.



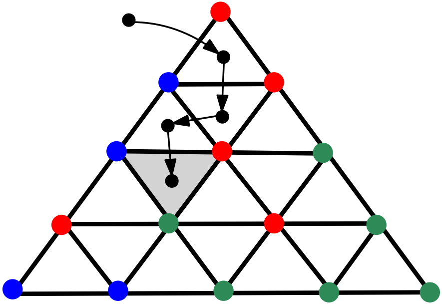
Sperner's lemma, proof



Problem `2D-Sperner`.
Input. A triangulation of a triangle; a Sperner labeling (described by a circuit)
Output. A small colorful triangle

`2D-Sperner` is in PPAD (Scarf 1967, Papadimitriou 1992).

Sperner's lemma, proof



Problem `2D-Sperner`.
Input. A circuit $\left\{(n_1, n_2, n_3) \in \mathbb{Z}_+^3 \colon n_1 + n_2 + n_3 = k\right\} \longrightarrow \{1, 2, 3\}$
Output. A small colorful triangle

`2D-Sperner` is in PPAD (Scarf 1967, Papadimitriou 1992).

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PPAD-complete

Theorem Papadimitriou 1992; Chen and Deng 2009

`2D-Sperner` is PPAD-complete.

The following problem is the first natural PPAD-complete problem.

Problem `Bimatrix`.
Input. Two $m \times n$ matrices P, Q
Output. A Nash equilibrium

Theorem Cheng and Deng 2009

`Bimatrix` is PPAD-complete.

PPAD-complete

Theorem Papadimitriou 1992; Chen and Deng 2009

`2D-Sperner` is PPAD-complete.

The following problem is the first natural PPAD-complete problem.

Problem `Bimatrix`.
Input. Two $m \times n$ matrices P, Q
Output. $x \in \triangle^m$, $y \in \triangle^n$ such that

- $x^\top P y \geqslant (x')^\top P y'$ for all $x' \in \triangle^m$
- $x^\top Q y \geqslant x^\top Q y'$ for all $y' \in \triangle^n$

Theorem Cheng and Deng 2009

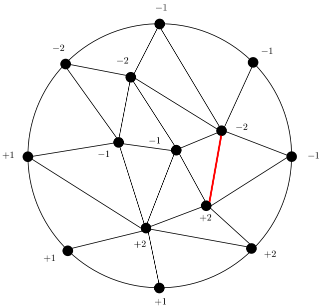
`Bimatrix` is PPAD-complete.

PPA-complete

Lemma Tucker 1946

Consider a triangulation of a disk, whose vertices are labeled with elements from $\{-1, +1, -2, +2\}$. Suppose that antipodal vertices from the boundary get opposite labels. Then there exists a complementary edge.

- in PPA (Freund, Todd 1981, Papadimitriou (?) 1992)
- PPA-complete (Aisenberg, Bonet, Buss 2015)

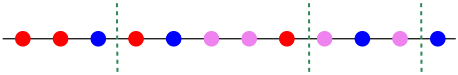


PPA-complete

Theorem Goldberg, West 1985

Given an open necklace with t types of beads and an even number of beads of each type, there exists a way to share it between two thieves with at most t cuts.

- in PPA (Papadimitriou 1992)
- PPA-complete (Filos-Ratzikas, Goldberg 2019)
- This is the unique natural PPA-complete problem known so far.



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The complexity class PPA-k

Problem `Bipartite-mod-`k.
Input. A bipartite graph; a vertex of degree $\neq 0 \pmod{k}$
Output. Another vertex of degree $\neq 0 \pmod{k}$

(The graph can be potentially huge; described by a circuit.)

> PPA-k = class of function problems that can be polynomially reduced to `Bipartite-mod-`k

Introduced by Papadimitriou (1992)

PPAD $\subseteq \bigcap_{k=2}^{\infty}$ PPA-k (Johnson 2011)

A PPA-p-complete problem

Let p be a prime number.

Problem `Chevalley−Warning`.
Input. $f_1, \ldots, f_m \in \mathbb{F}_p[X_1, \ldots, X_n]$ such that $\sum_i \deg(f_i) < n$; a common zero
Output. Another common zero

Theorem Göös, Kamath, Sotiraki, Zampetakis 2020

`Chevalley−Warning` (in a slightly more general version) is PPA-p-complete.

This is the unique natural PPA-p-complete problem known so far.

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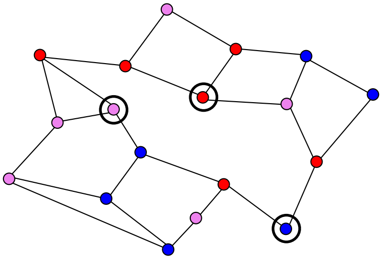
Theorem Haxell 1995

In a colored graph with maximum degree Δ, there always exists an independent set intersecting every color class of size at least 2Δ.

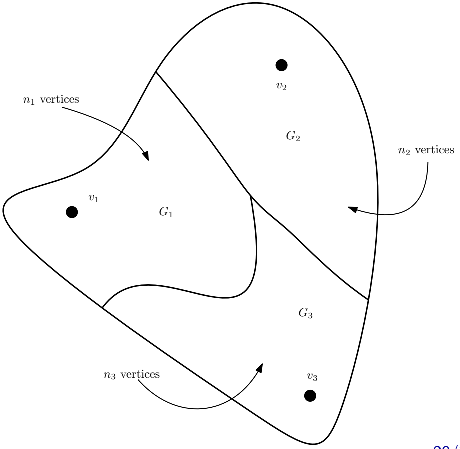
- Haxell's original proof is elementary; proof based on a Sperner-type argument (Aharoni, Berger, Ziv 2007)
- Polynomial algorithms for weaker versions
- Questions: In PPAD? Hard?



Theorem Győri 1976, Lovász 1977

Let G be a k-connected graph, $v_1, v_2, \ldots, v_k$ distinct vertices, and $n_1, n_2, \ldots, n_k$ positive integers with $n_1 + n_2 + \cdots + n_k = |V(G)|$. Then G has disjoint connected subgraphs $G_1, G_2, \ldots, G_k$ such that $|V(G_i)| = n_i$ and $v_i \in V(G_i)$.

- Győri's proof is elementary; Lovász's proof relies on topology
- Polynomial for $k \leqslant 3$ (Wada and Kawaguchi 1993)
- Questions: In PPAD? Hard?



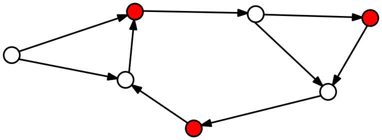
A simple directed graph is clique-acyclic if every clique is acyclic.

kernel = subset of vertices that is independent and absorbing

Theorem Boros, Gurvich 1996

Every clique-acyclique orientation of a perfect graph has a kernel.

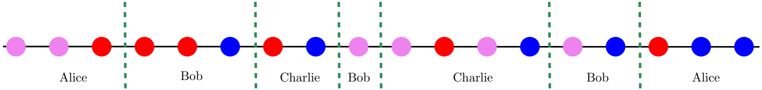
- All known proofs rely on topology
- Simple proof based on Sperner's lemma applied on the polar of $P = \mathrm{STAB}(G) - \mathbb{R}^n_+$ (Király, Pap 2009)
- Questions: In PPAD? Hard?
- Issue: P uses matrices with rows indexed by all (maximal) cliques



Theorem Alon 1987

Given an open necklace with t types of beads and a multiple of k beads of each type, there exists a way to share it between k thieves with at most $t(k-1)$ cuts.

- in PPA-p (Filos-Ratzikas, Hollender, Sotiraki, Zampetakis 2021) when k is a prime number p
- Hard?



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Takeaways

- PPA, PPAD, etc. = complexity classes useful to prove hardness results for computing objects of existential results (TFNP)
- a way to "sell" constructive proofs
- at the intersection of mathematics and computer science (and see next slide!)
- many challenging open problems

Strength of theorems

- `2D-Sperner` is PPAD-complete
- `2D-Tucker` is PPA-complete
- PPA $\neq$ PPAD (strongly believed)

Conclusion: you cannot prove Tucker from Sperner.

Complexity theorems provide a framework where the questions like

- are these theorems equivalent?
- can you deduce this theorem from this one?

make complete sense.

Computer scientists answer questions from pure mathematicians.

E.g., the Borsuk–Ulam theorem cannot be deduced from the Hairy Ball theorem (Goldberg, Hollender 2019).

Thank you