Logarithmic space, PCP and the hardness of approximation

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- you can store a constant number of names of vertices in memory
- you can store a logarithmic number of boolean values
- you can store a constant number of polynomially bounded integers (for instance counters).

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(it can be proved that in $O(n^3)$ steps, the random walk has visited all the vertices of the connected component of s with high probability).

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If $D \geq 2$, for each vertex w, we test whether $d(u, w) \leq \lfloor D/2 \rfloor$ and $d(v, w) \leq \lceil D/2 \rceil$ (one after the other) and if both are true we answer yes.

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The space needed is $S(D) \leq O(\log n) + S(D/2)$, and thus

$$S(D) = O((\log n)(\log D)).$$

In particular, testing whether two vertices are at distance at most n takes $O(\log^2 n)$ space.

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For technical reasons we then attach self-loops to each vertex, turning G into a d-regular graph for some fixed d (constant, but not too small).

Expanders

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Example. for $p \to \infty$, p prime, look at the graph with vertex set $\mathbb{Z}_p$, in which each x is adjacent to $x+1$, $x-1$, and x^{-1}.

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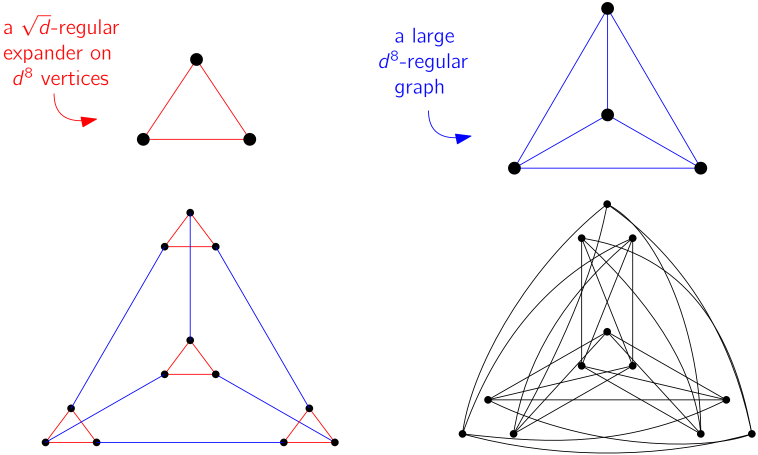
- take the zig-zag product of our current graph with a small (fixed) expander. The expansion decreases slightly but the degree goes back to d.

The zig-zag product



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After $\log n$ iterations the graph still has size polynomial in n, and it is a (c, d)-expander for some constant c. So the graph has logarithmic diameter and we can solve s, t-connectivity in this graph in logarithmic space.

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- Given a graph, is there a cycle containing a given edge?
- Is a given graph bipartite?
- Do two graphs have the same number of connected components?
- Does a graph have an even number of connected components?

Conclusion on L

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- It is not known whether $RL = L$ (i.e. whether randomized logarithmic space can be completely derandomized).

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- The prover presents a certificate of polynomial size, and the verifier only checks the constant number of locations it chose, and based only on this constant number of bits, decides to accept or reject the proof.
- If the instance is positive, then the verifier must accept, and if the instance is negative, then the verifier must reject with probability at least $\frac{1}{2}$, for any certificate.

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(by c-approximation algorithm, we mean an algorithm that returns a value between $\omega(G)/c$ and $\omega(G)$.)

Connection between the two versions

Recall that in 3-SAT, we have a formula of the type $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_4 \vee \neg x_5) \wedge \ldots$, where there are 3 literals per clause and the x_i's are boolean variables, and the goal is to decide whether the formula can be satisfied (so all clauses have to be satisfied) by some assignment of values to the x_i's.

Connection between the two versions

The clique approximation version of the PCP theorem is equivalent to the statement that there is some $\varepsilon > 0$ such that if we are given a 3-SAT formula in which

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Now any problem in NP can be reduced to this gap version of 3-SAT, and this version has a simple probabilistically checkable certificate, consisting of the values of the variables x_i.

The verifier just checks a constant number of clauses selected uniformy at random and says that the formula is satisfiable if all these clauses are satisfied.

Gap amplification

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In particular, we cannot approximate the clique number in polynomial time within a factor $c < b/a$ unless $\mathrm{P} = \mathrm{NP}$.

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Now any c_1-approximation algorithm $\mathcal{A}_1$ for the clique number can be turned into a c_2-approximation algorithm $\mathcal{A}_2$ with $1 < c_2 < c_1$ by defining $\mathcal{A}_2(G) = (\mathcal{A}_1(G * k))^{1/k}$ for sufficiently large (but constant) k, because:

$$\omega(G)/c_2 \leq \omega(G)/c_1^{1/k} = (\omega(G * k)/c_1)^{1/k} \leq \mathcal{A}_2(G) \leq \omega(G),$$

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Let G be an n-vertex graph, and let F be a (c, d)-expander graph on the same vertex set as G. For some integer t, we define a new graph H whose vertices are the $t+1$-vertex walks $x_0, \ldots, x_t$ in F, and in which two walks are adjacent if and only if their union is contained in some clique of G.

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With $t = \log n$, the ratio between the two bounds is close to $(b/a)^t$ which is polynomial in n (and N).

Random walks in expander graphs

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Random walks in expander graphs

Let G be a (c, d)-expander graph on n vertices.
Suppose we take a vertex x_0 uniformly at random in G, and then perform a random walk $x_0, \ldots, x_t$ of length t starting at x_0.
Then for any subset $S \subseteq V(G)$, the probability that $x_0, x_1, \ldots, x_t$ are all in S is at most $(|S|/n + o(1))^t$ and at least $(|S|/n - o(1))^t$.