Szemerédi's graph regularity lemmas

Paul Bastide

University of Oxford

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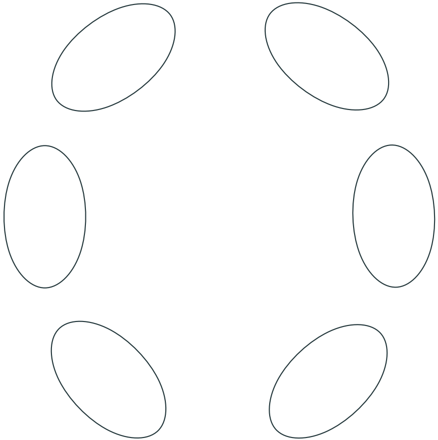

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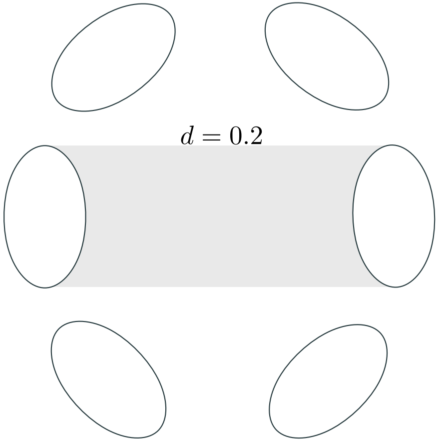

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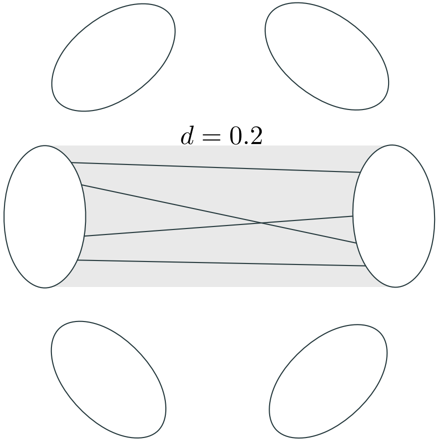

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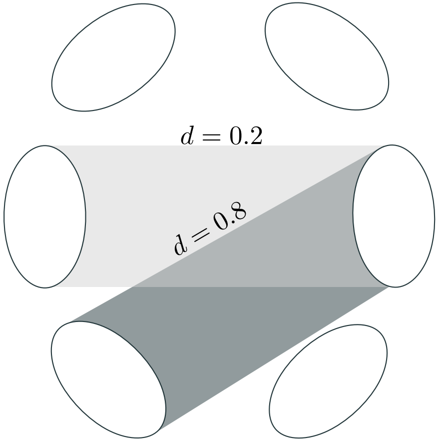

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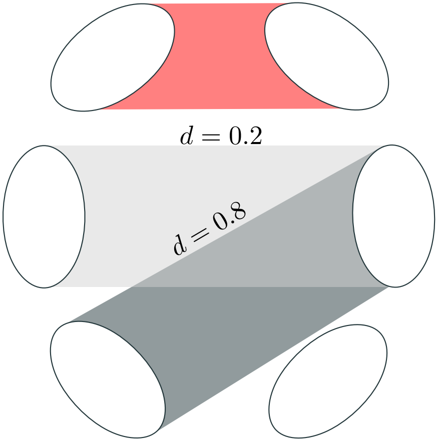

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Number of edges

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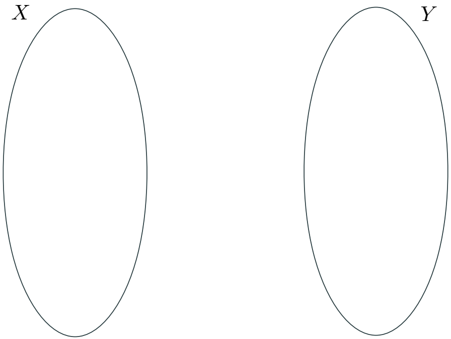

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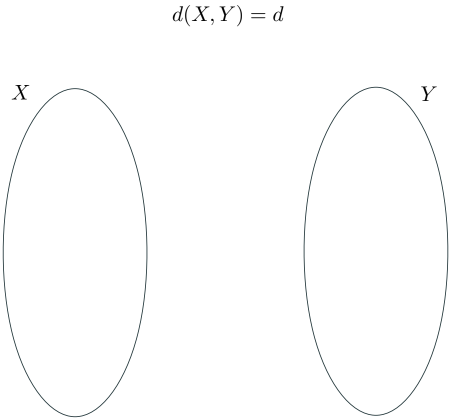

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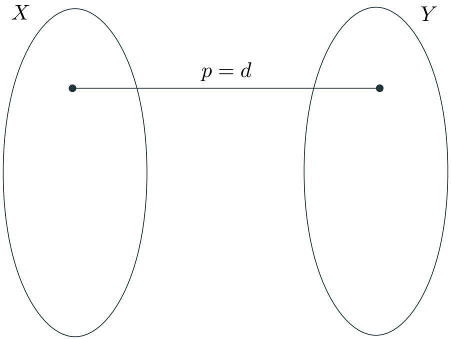

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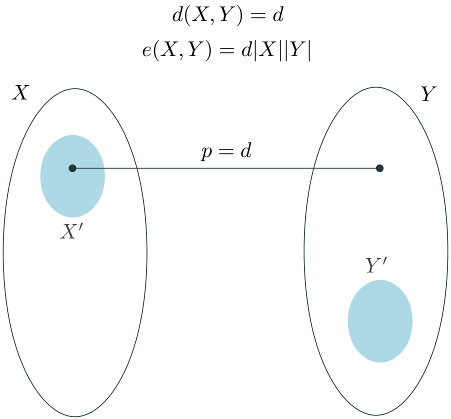

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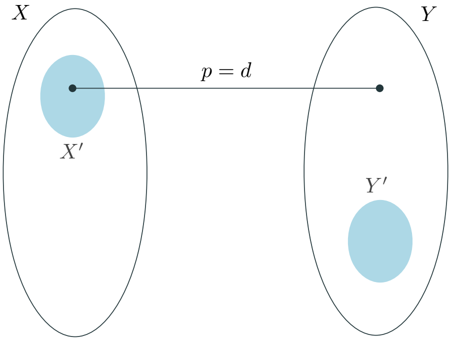

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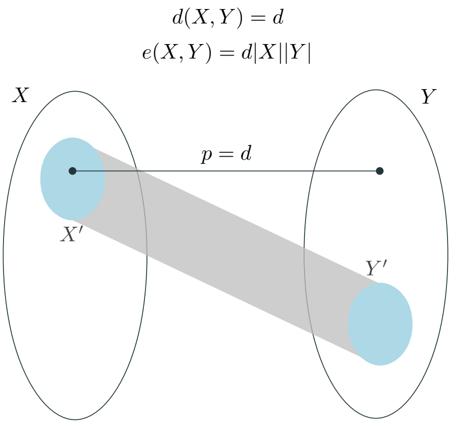

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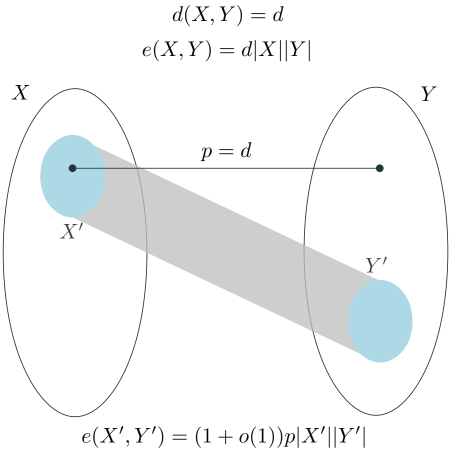

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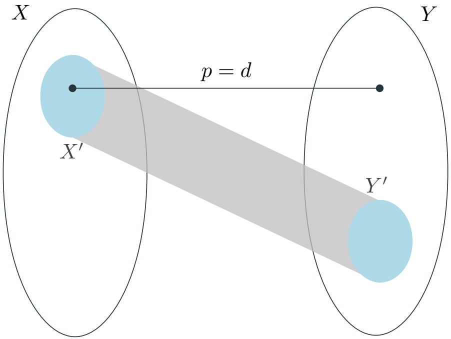

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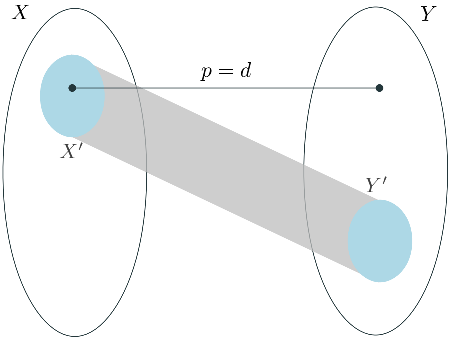

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Witness of non-regularity

- If $d(X', Y') \neq d(X, Y) \pm \varepsilon$

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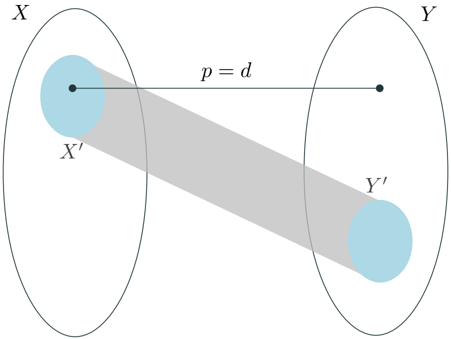

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The lemma

ε-regular partition

- $V(G) = V_1 \sqcup \ldots \sqcup V_M$, where $\displaystyle\sum_{(V_i, V_j) \text{ not } \varepsilon\text{-regular}} |V_i||V_j| \leq \varepsilon n^2$.

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For every $\varepsilon > 0$, there exists M such that every graph has an ε-regular partition of size M.

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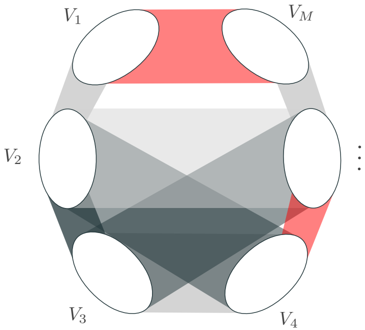

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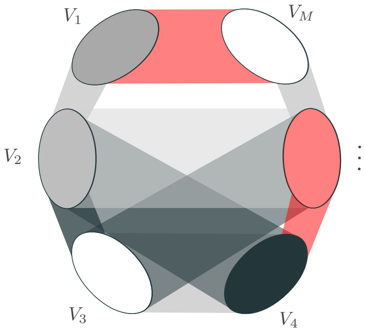

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We design an energy function q s.t.

$$q(\mathcal{P}) \in [0, 1]$$

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Input: A graph G

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While $\mathcal{P}$ is not ε-regular:

- For each pair $(V_i, V_j) \in \mathcal{P}^2$ that is not ε-regular.
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Refine $\mathcal{P}$ with all $(A_{i,j})$.

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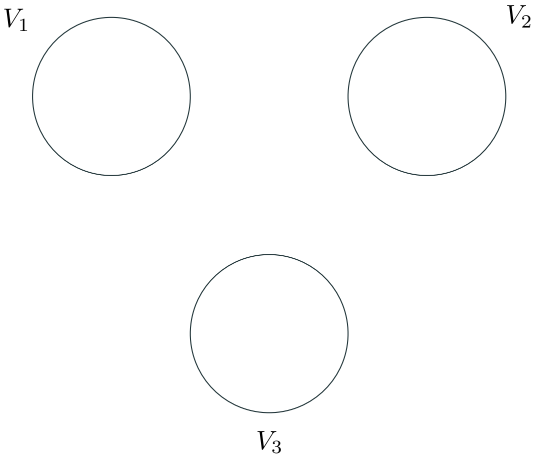

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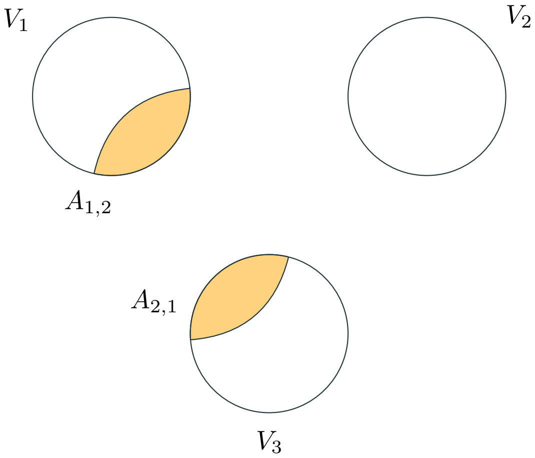

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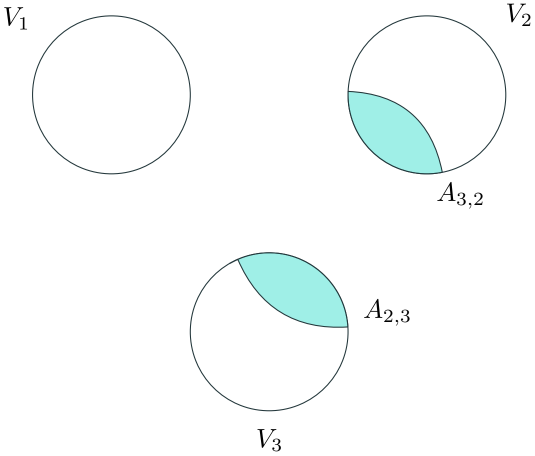

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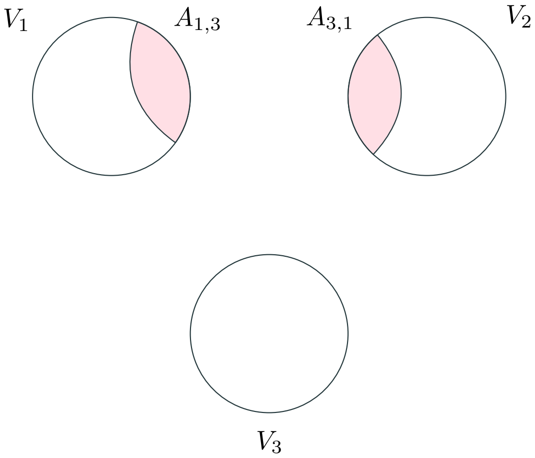

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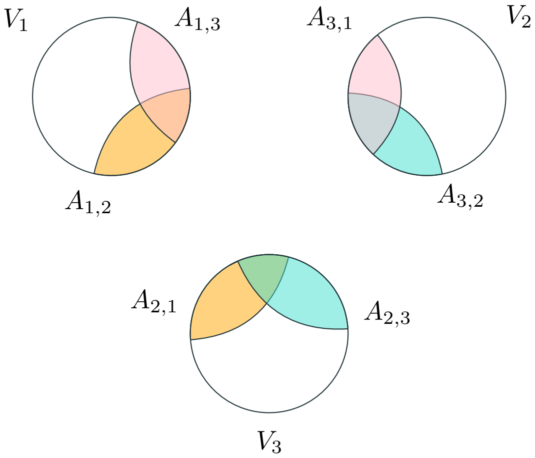

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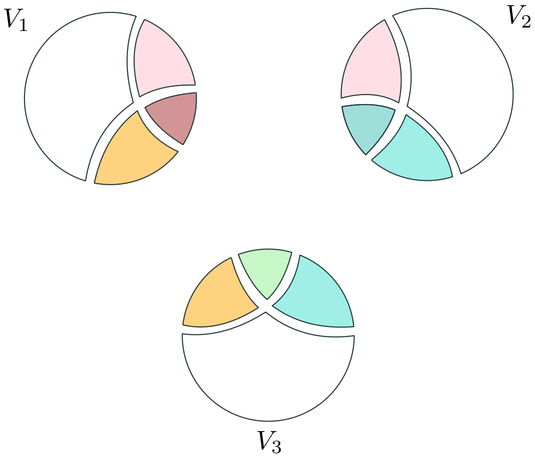

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Monotonicity

q does not decrease under refinement.

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Constant increase

Let $\mathcal{Q}$ be the refinement obtained after one iteration of the "While" loop. If $\mathcal{Q}$ is not ε-regular, then

$$q(\mathcal{Q}) \geq q(\mathcal{P}) + \varepsilon^5$$

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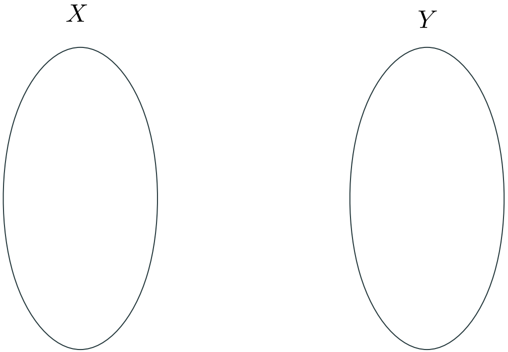

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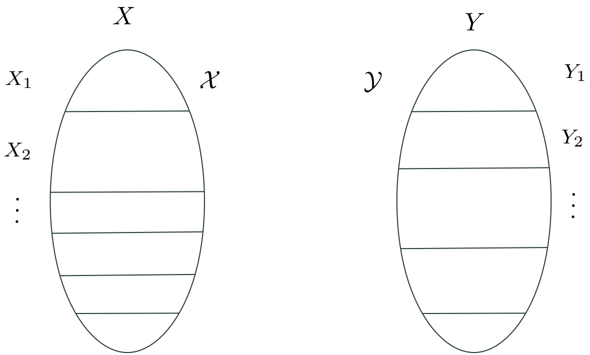

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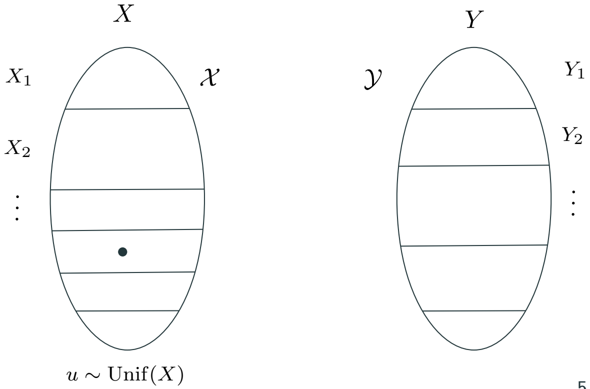

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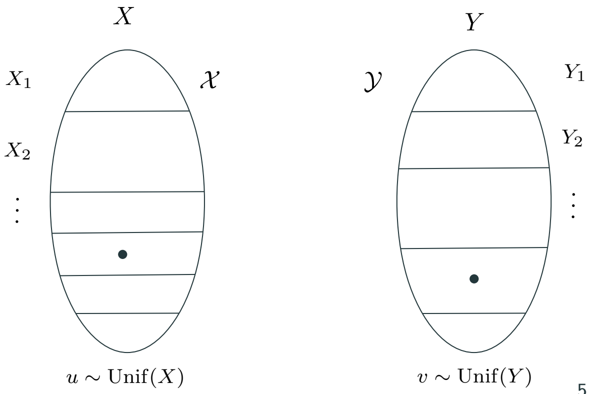

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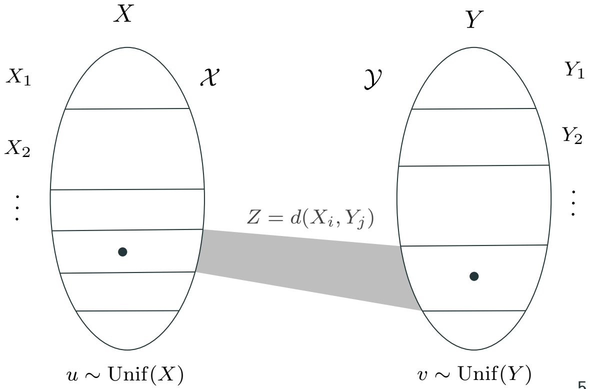

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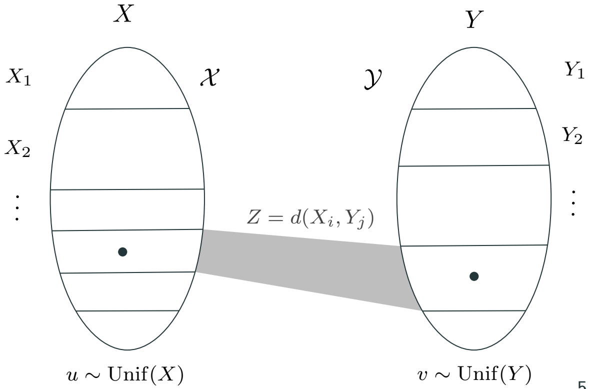

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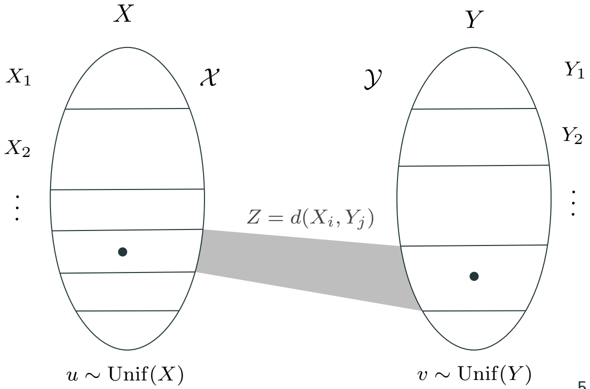

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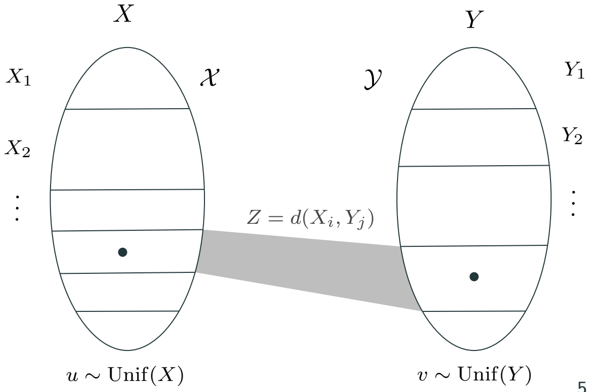

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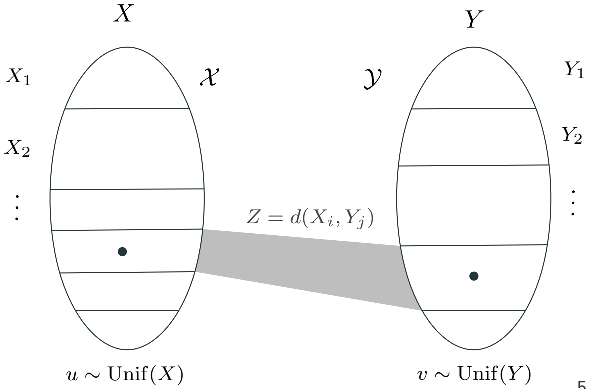

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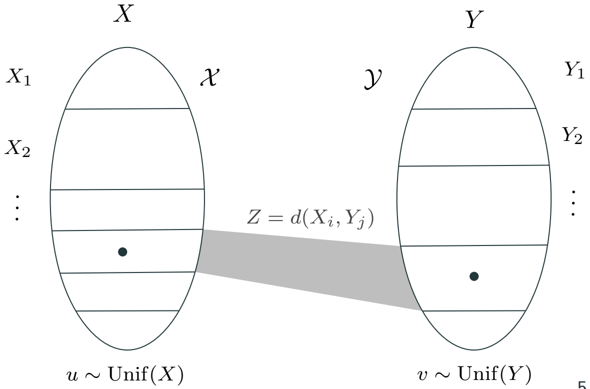

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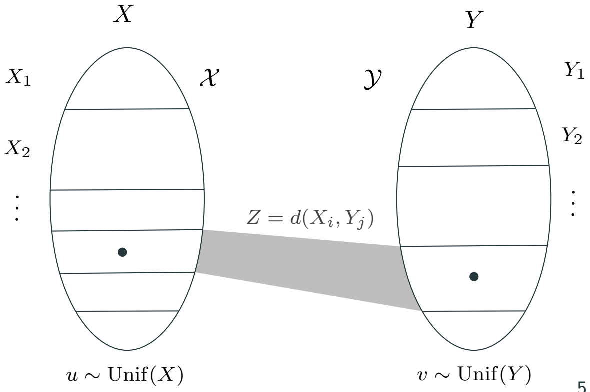

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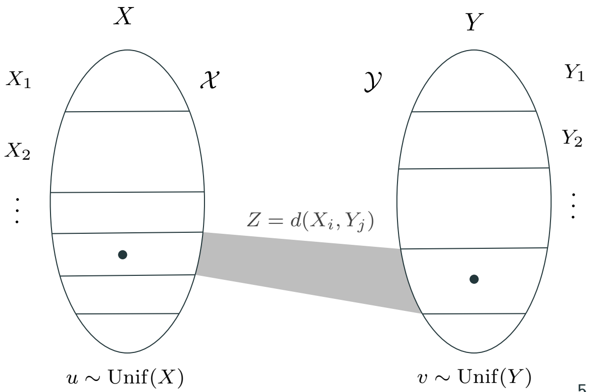

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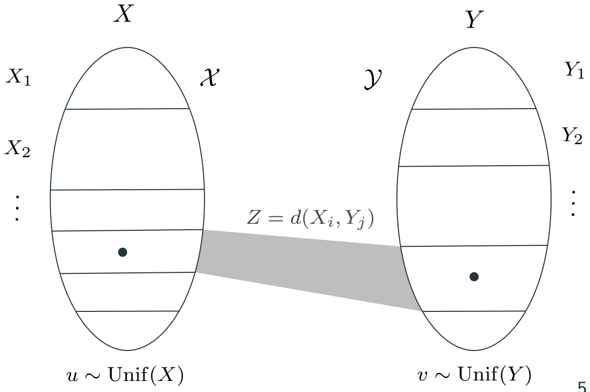

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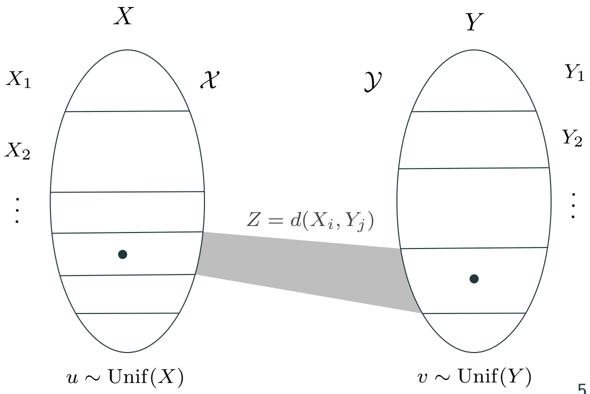


The lemma

ε-regular pair

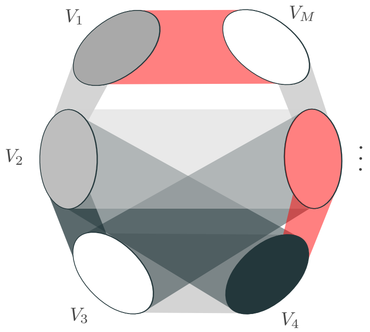
- All $X' \subseteq X$ and $Y' \subseteq Y$ such that $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$ satisfy $d(X', Y') = d(X, Y) \pm \varepsilon$

ε-regular partition

- $V(G) = V_1 \sqcup \ldots \sqcup V_M$, where $$\sum_{(V_i, V_j) \text{ not } \varepsilon\text{-regular}} |V_i||V_j| \leq \varepsilon n^2.$$

Szemerédi's graph regularity lemma

For every $\varepsilon > 0$, there exists M such that every graph has an ε-regular partition of size M.



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4) What about sparse graphs? (graphs with $o(n^2)$ edges)

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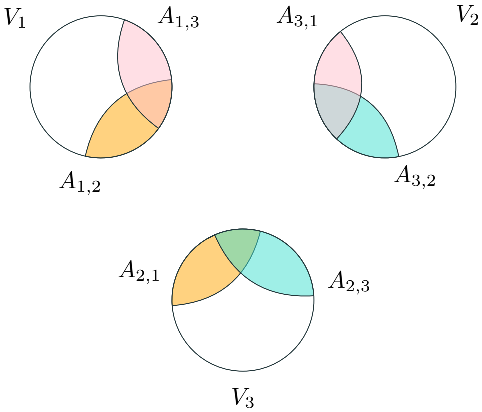

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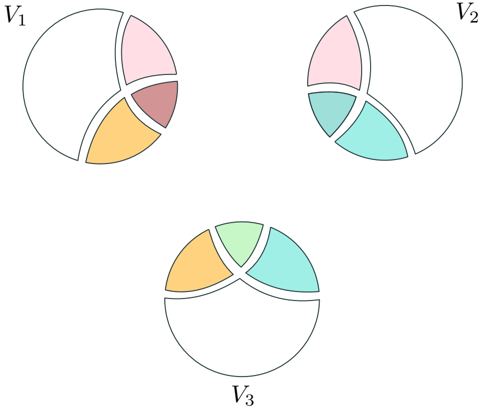

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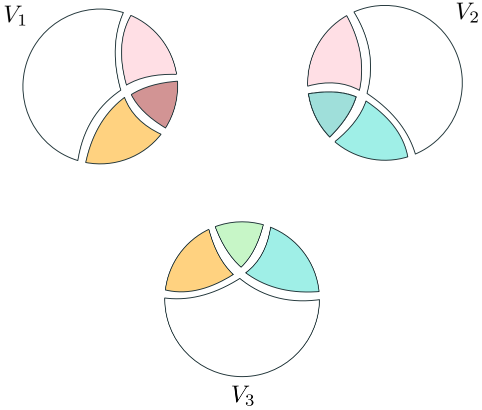

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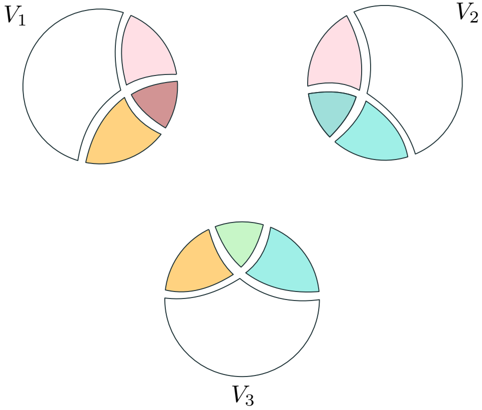

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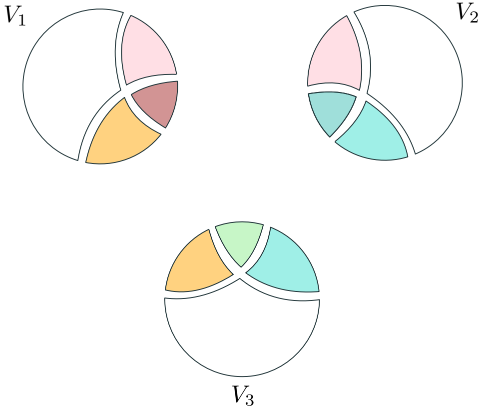
$\exists c > 0$, such that $\forall \varepsilon > 0$ small enough, there exists a graph such that any ε-regular partition contains $\text{tower}(2, \varepsilon^{-c})$ different parts.



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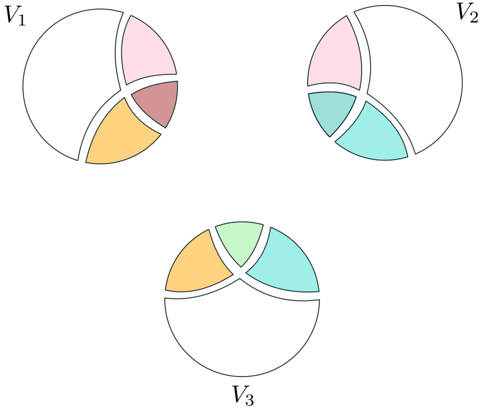
Maybe try your best not to use it!



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For any $\varepsilon, C > 0$, there exists M such that for any graph G and any partition $\mathcal{P}$ of $V(G)$ with $|\mathcal{P}| = C$, there exists a refinement of $\mathcal{P}$ that is ε-regular and has M parts.

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Dream

For any $\varepsilon > 0$, there exists M such that for any graph G, there exists a set $X \subseteq V(G)$ with $|X| = \varepsilon |V(G)|$ such that $G - X$ admits a partition $\mathcal{P}$ where every pair is ε-regular.

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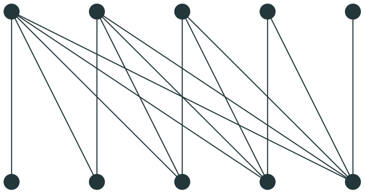
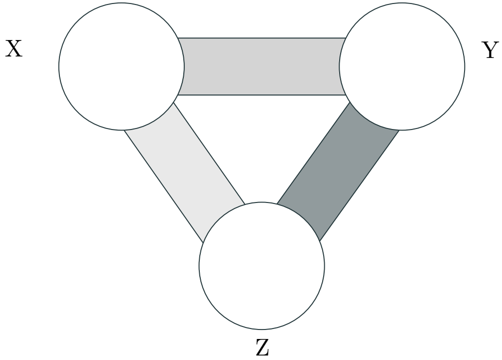


Triangle Counting Lemma



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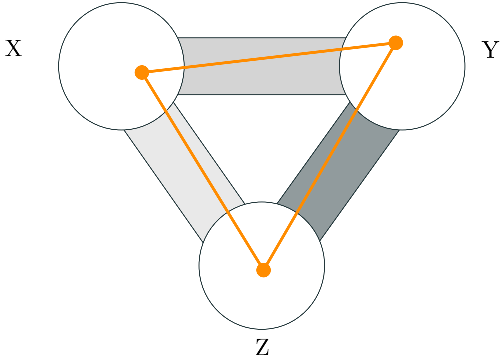

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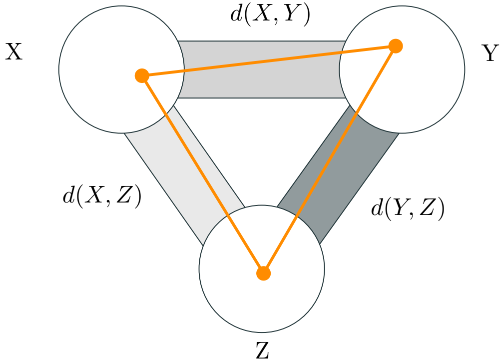

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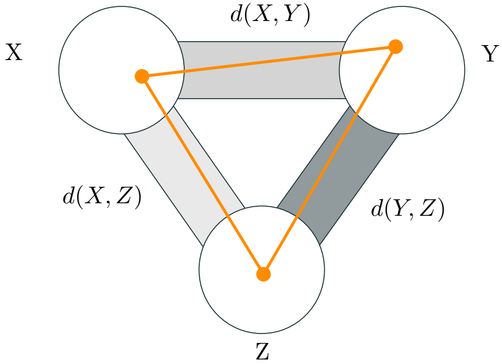

$\#\{XYZ\text{–triangle}\}$

$\approx |X||Y||Z|d(X,Y)d(Y,Z)d(X,Z)$

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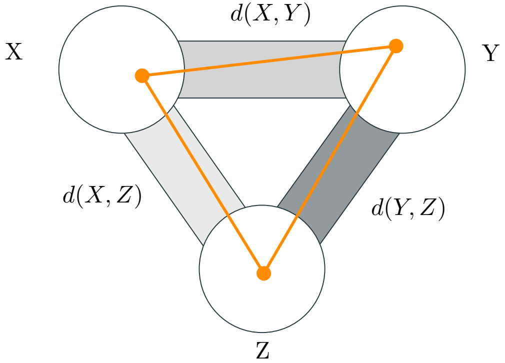

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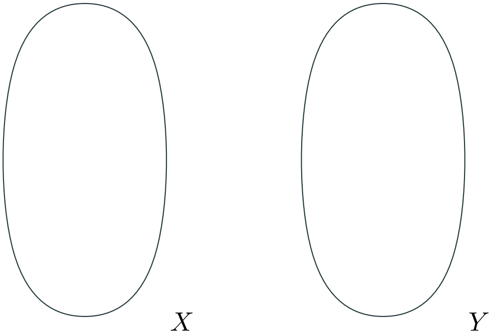

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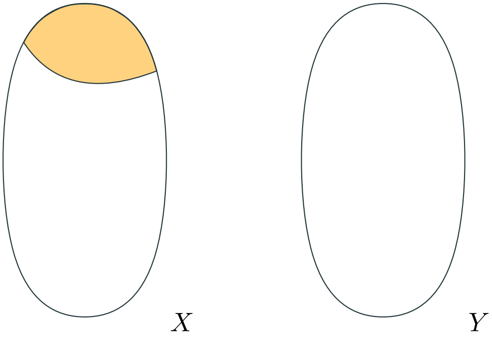

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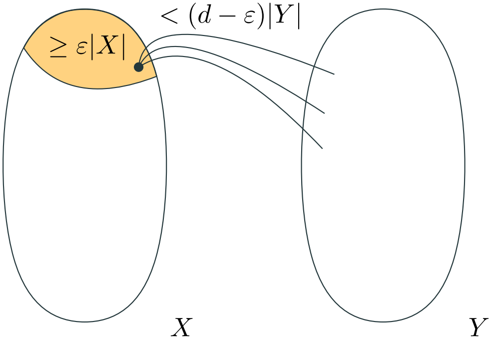

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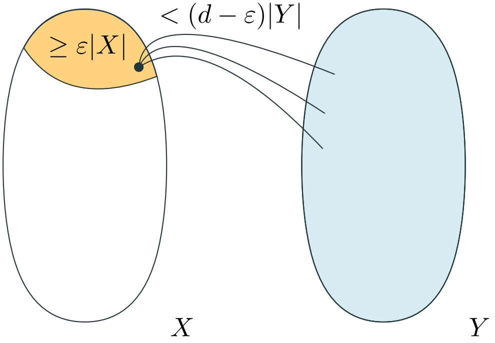

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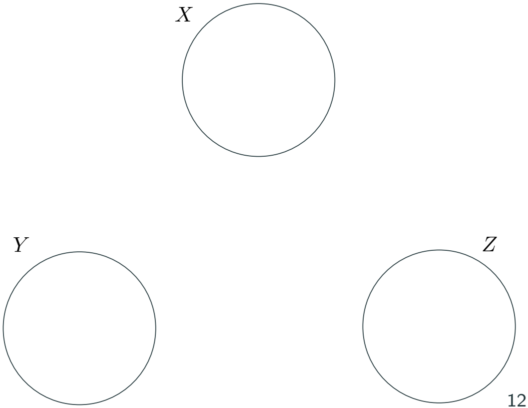

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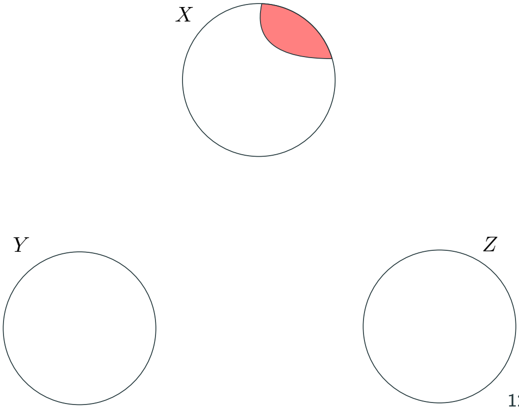

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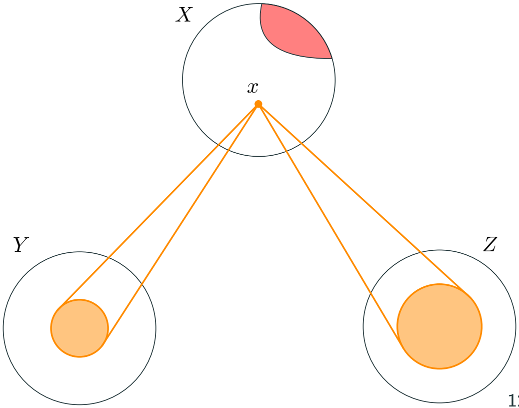

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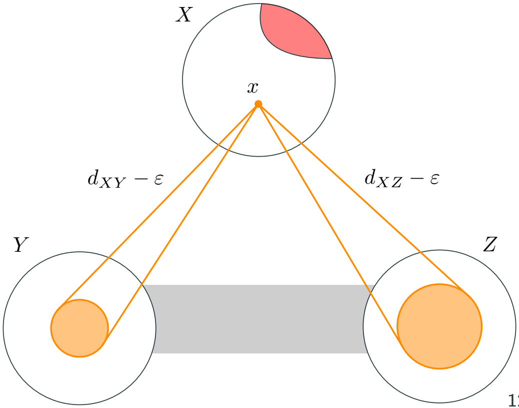

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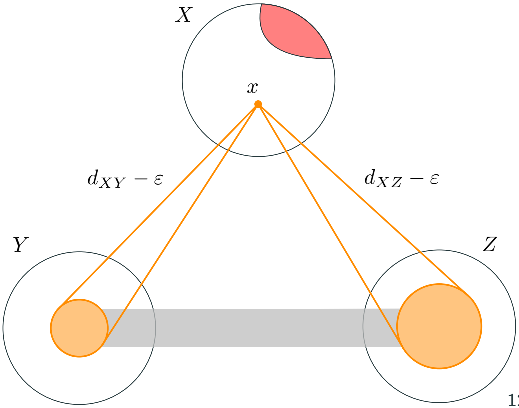

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$X, Y, Z \subseteq G$ with (X, Y) ε-regular and $d_{XY} = d(X, Y) \geq 2\varepsilon$ (similarly for (Y, Z) and (X, Z)), then

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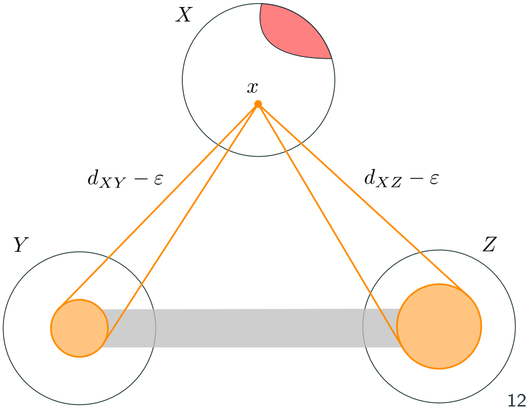

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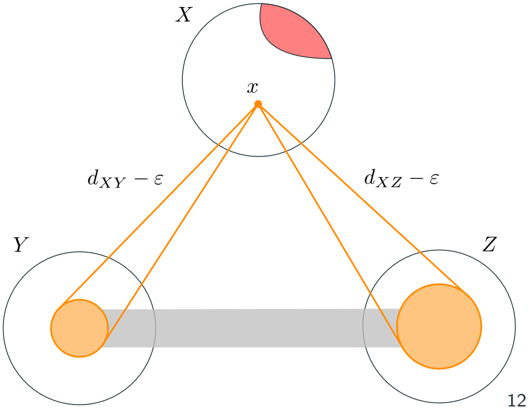

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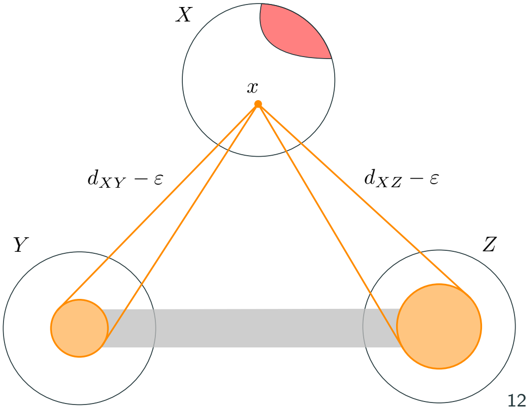

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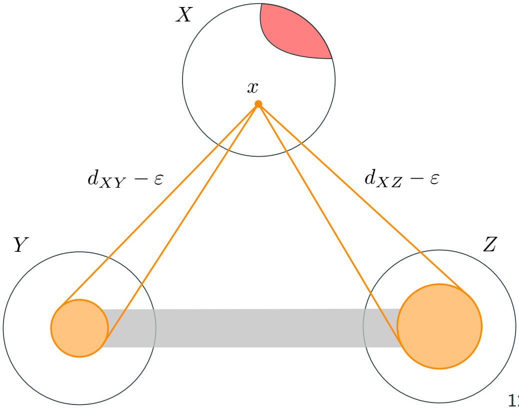

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Any set $A \subseteq [n]$ that avoids $x, x + a, x + 2a$ for $x, a \in [n]$ has size $o(n)$.

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What about other graphs then the triangle?

H-Embedding Lemma

Consider H r-partite, max degree Δ. Suppose $V_1, \ldots, V_r \subseteq V(G)$ are pairwise ε-regular, with $\forall i \in [r], |V_i| \geq |V(H)|/\varepsilon$ and all densities $\geq 2\varepsilon^{1/\Delta}$ then G contains a copy of H.

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Let H with $V(H) = [k]$, $\forall \varepsilon > 0$, and $V_1, \dots, V_k \subseteq G$ all pairwise ε-regular, then the number of $(v_1, v_2, \dots, v_k) \in V_1 \times \cdots \times V_k$ such that if $ij \in E(H)$ then $v_i v_j \in E(G)$ is

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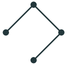
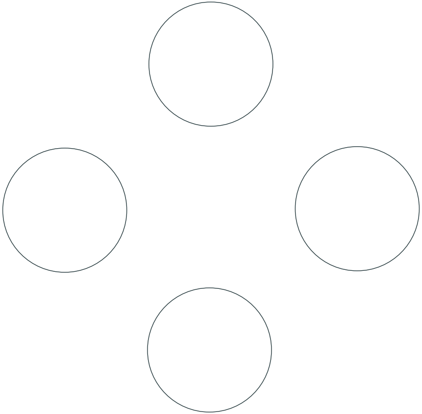
$H =$ 



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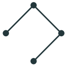
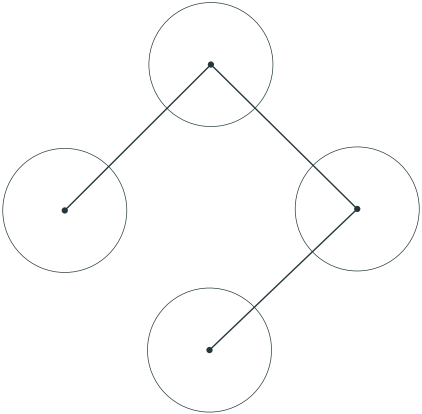
$H =$ 



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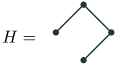

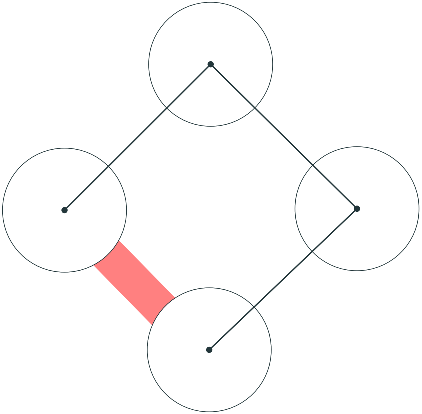

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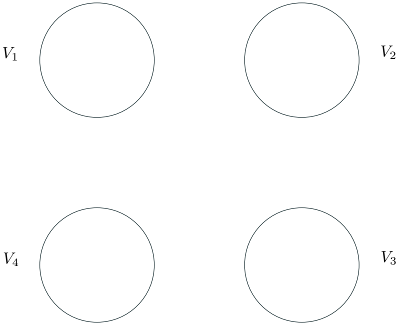

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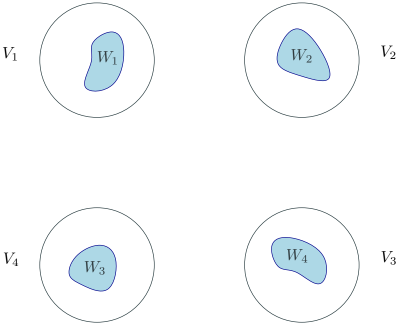

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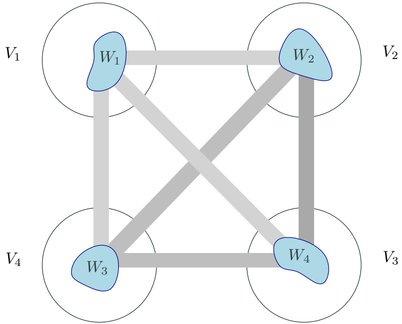

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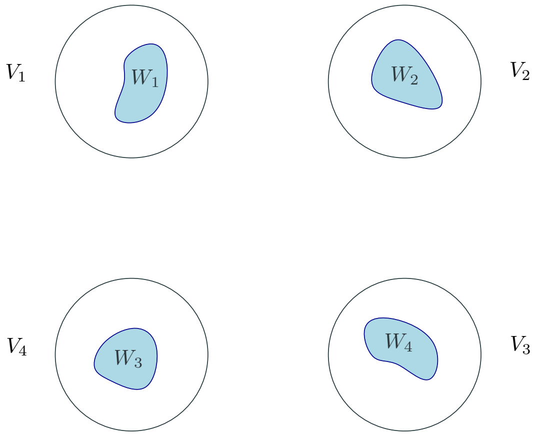

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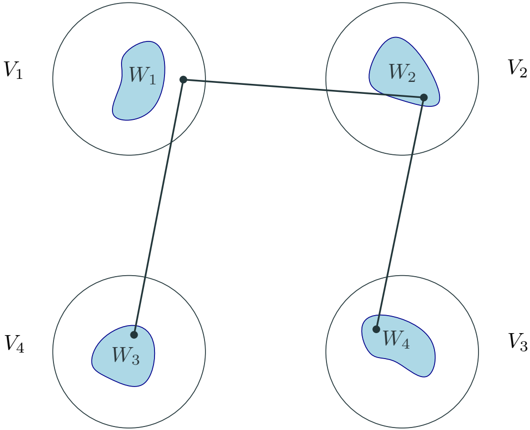

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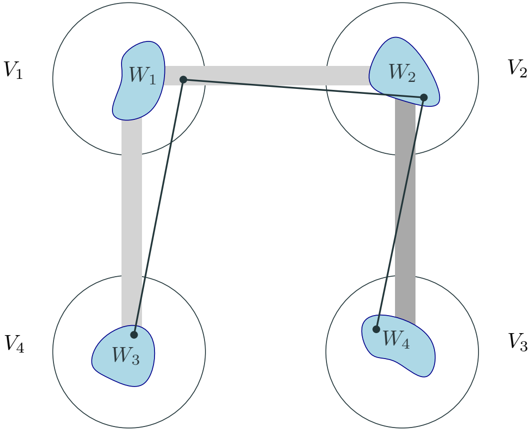

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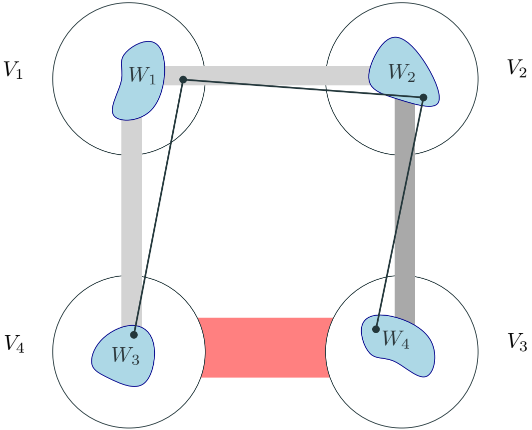

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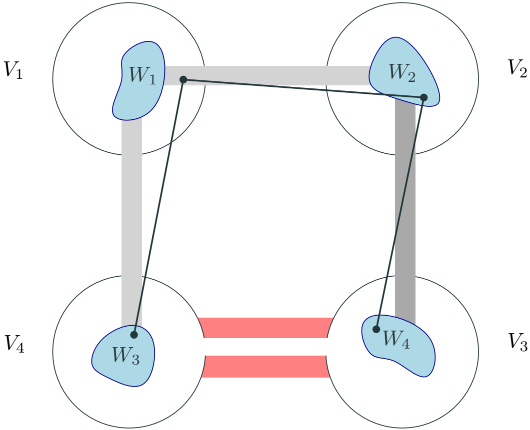

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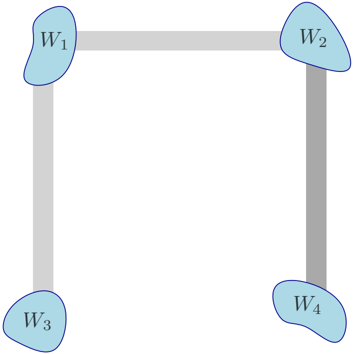

H-induced removal Lemma

$\forall H$, $\varepsilon > 0$ $\exists \delta > 0$ such that any graph G with $< \delta n^{|V(H)|}$ induced copies of H can be made induced H-free by removing or adding $\leq \varepsilon n^2$ edges.

1) Partition: Apply strong regularity lemma, obtain $W_i \subseteq V_i$.
2) Clean: For all pairs (i, j)
   - If $d(W_i, W_j) \leq \varepsilon$ remove all edges from V_i to V_j.
   - If $d(W_i, W_j) \geq 1 - \varepsilon$ add all edges from V_i to V_j.
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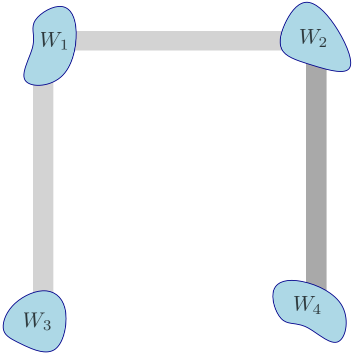
   Then $(W_i)_{i \in [k]}$ ensures that there are at least $f(\varepsilon) n^{|V(H)|}$ copies.



Infinite Removal Lemma

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For any class (possibly infinite) $\mathcal{H}$ of graphs, and any $\varepsilon > 0$, there exists $h_0, \delta > 0$ such that every n-vertex graph G with $< \delta n^{|V(H)|}$ induced copies of H for all $H \in \mathcal{H}$ with $|V(H)| \leq h_0$ can be made $\mathcal{H}$-free by adding/removing $< \varepsilon n^2$ edges.

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We say that "being triangle-free" is testable.

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Merci!

Sparse Regularity Lemma

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The Regularity lemma says nothing about sparse graphs! The trivial partition is already ε-regular with density 0.

Theorem (Scott 2010)

For every $\varepsilon > 0$ and $m \geq 1$, there exists $M = M(\varepsilon, m)$ such that every graph G with $|V(G)| \geq M$ has a balanced partition $V(G) = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_k$, with $|V_0| < \varepsilon |V(G)|$ and $m \leq k \leq M$, for which all but at most εk^2 pairs (V_i, V_j) are (ε)-regular, i.e.

$$|d(X', Y') - d(V_i, V_j)| < \varepsilon d(G)$$

whenever $X' \subseteq V_i$, $Y' \subseteq V_j$, $|X'| \geq \varepsilon |V_i|$, and $|Y'| \geq \varepsilon |V_j|$.