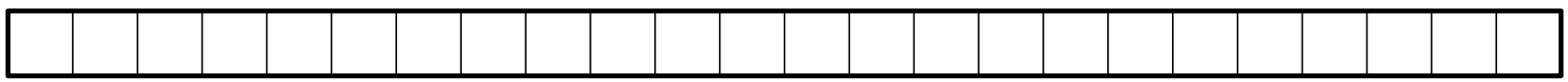
Monochromatic corners and Number-On-Forehead complexity

Julien Duron
(University of Warsaw, Supported by ERC BUKA)

Roth's theorem

$[N]$



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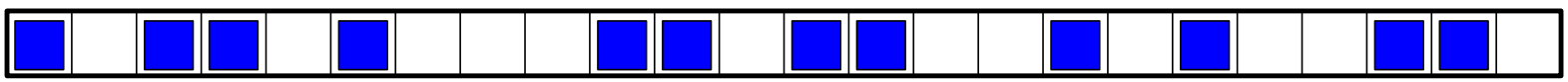
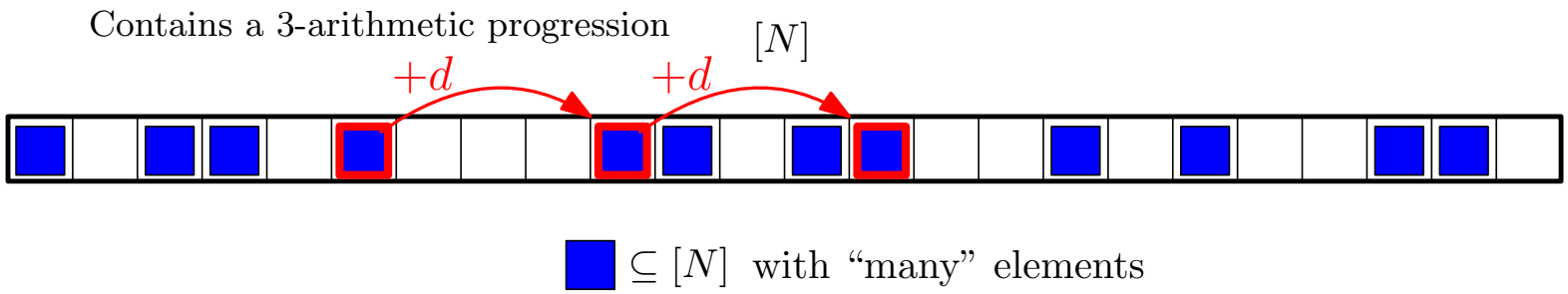


$\subseteq [N]$ with "many" elements

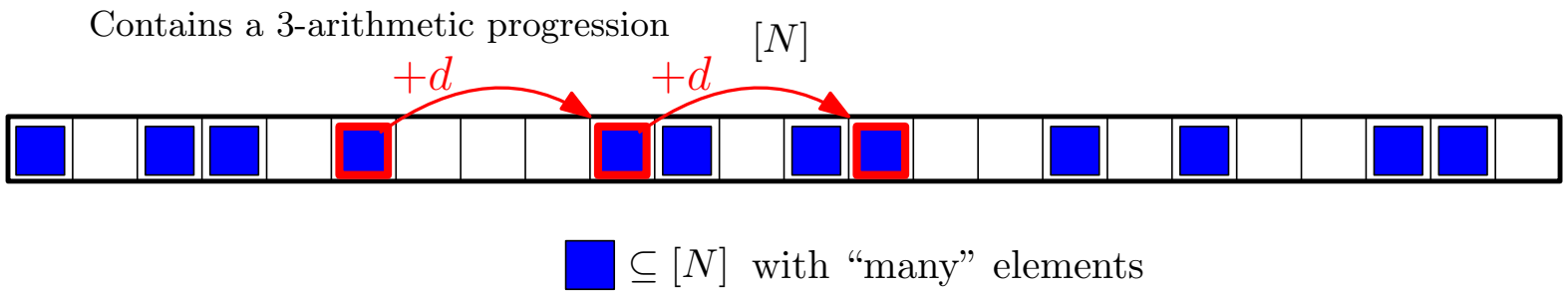
Roth's theorem



Roth's theorem

k-Arithmetic progression

$x, x + d, x + 2d, \ldots, x + (k - 1) \cdot d$ for some $x, d \in \mathbb{N}^+$.



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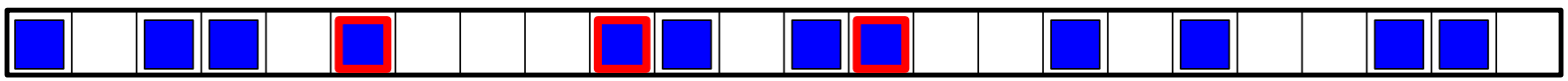

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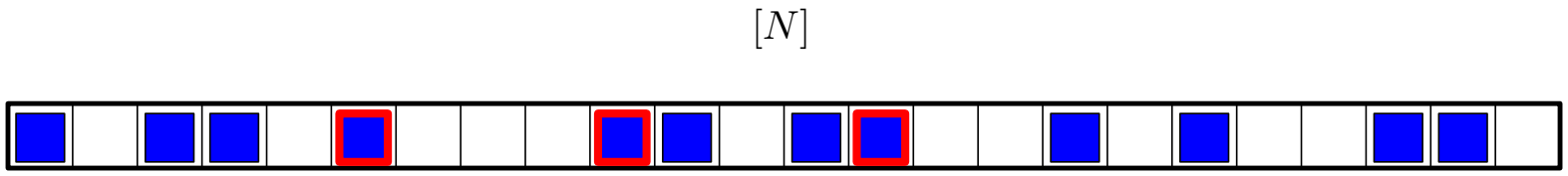
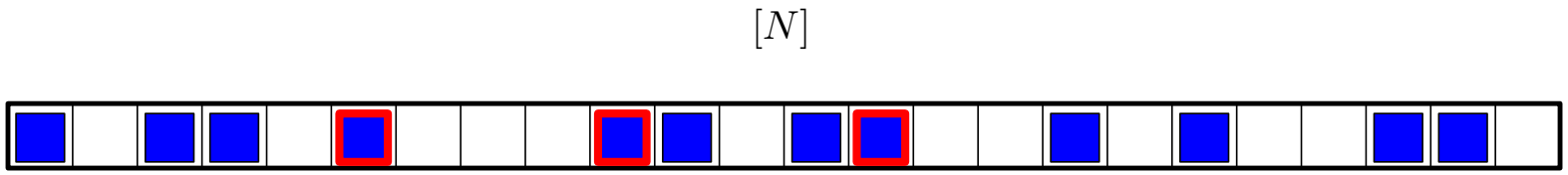


Density

The *density* of S in T is $|S|/|T|$.

Roth's theorem



Density

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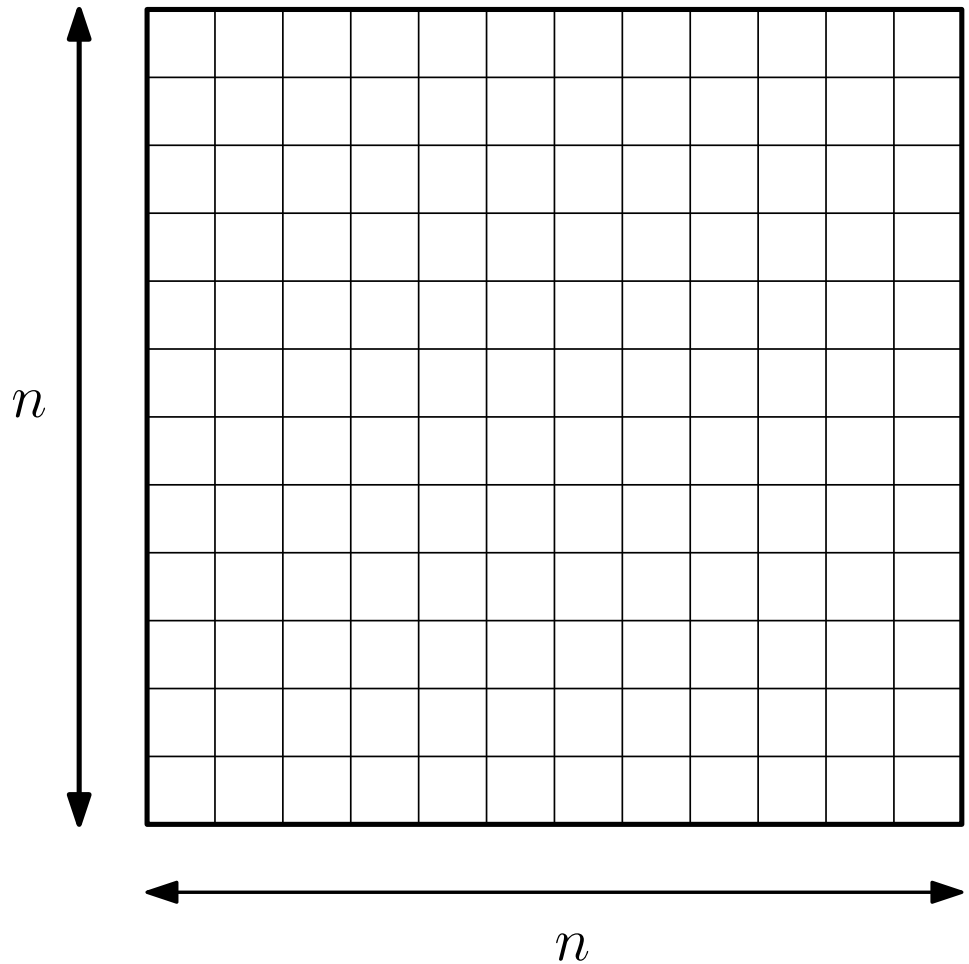
Roth's theorem, 1953

$\forall \epsilon > 0$, there is an N such that
Any $S \subset [N]$ with density $\geqslant \epsilon$ has a 3-arithmetic sequence.

Corners in $[n]^2$: definition



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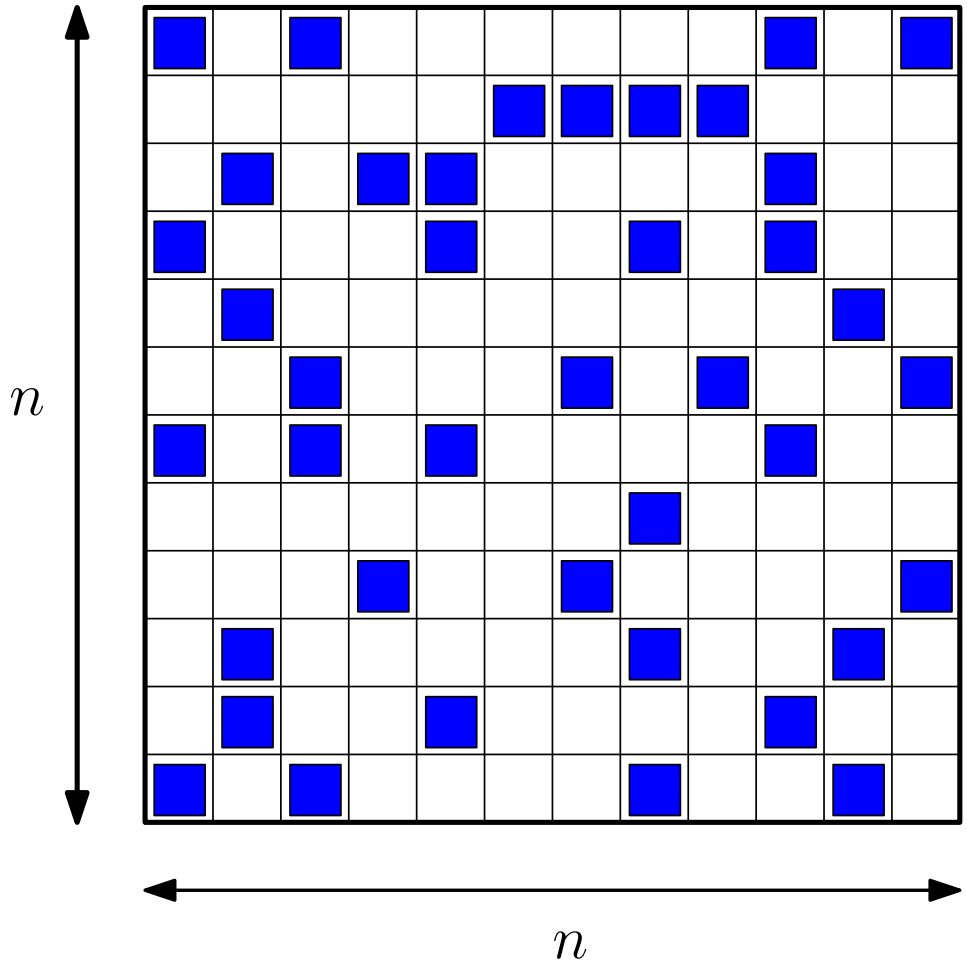

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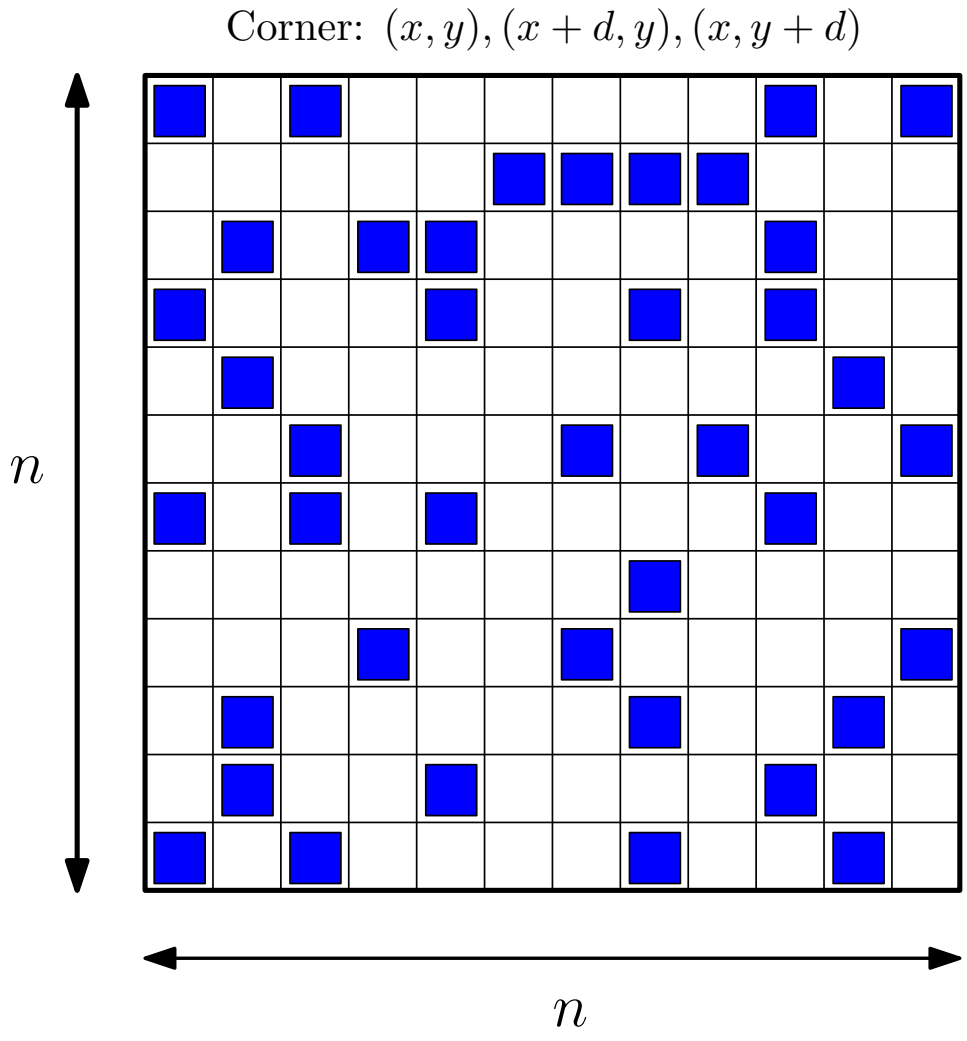

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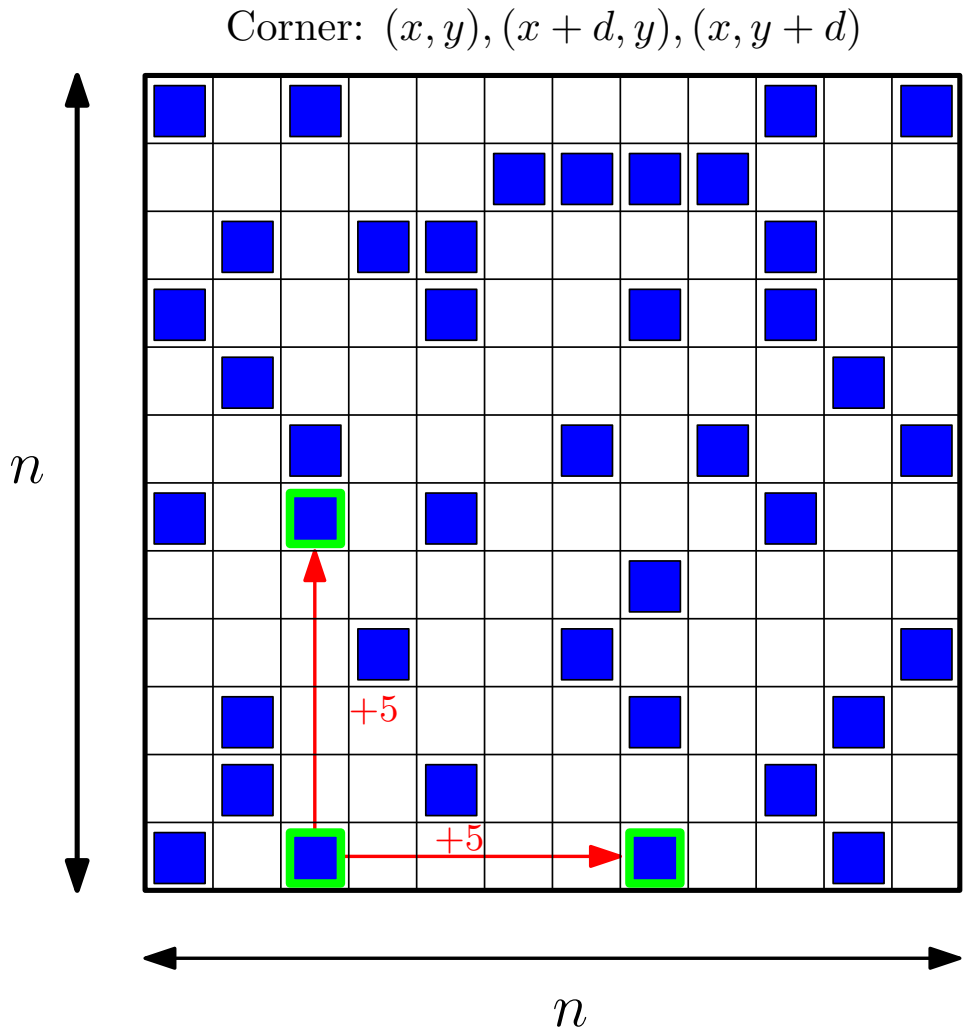

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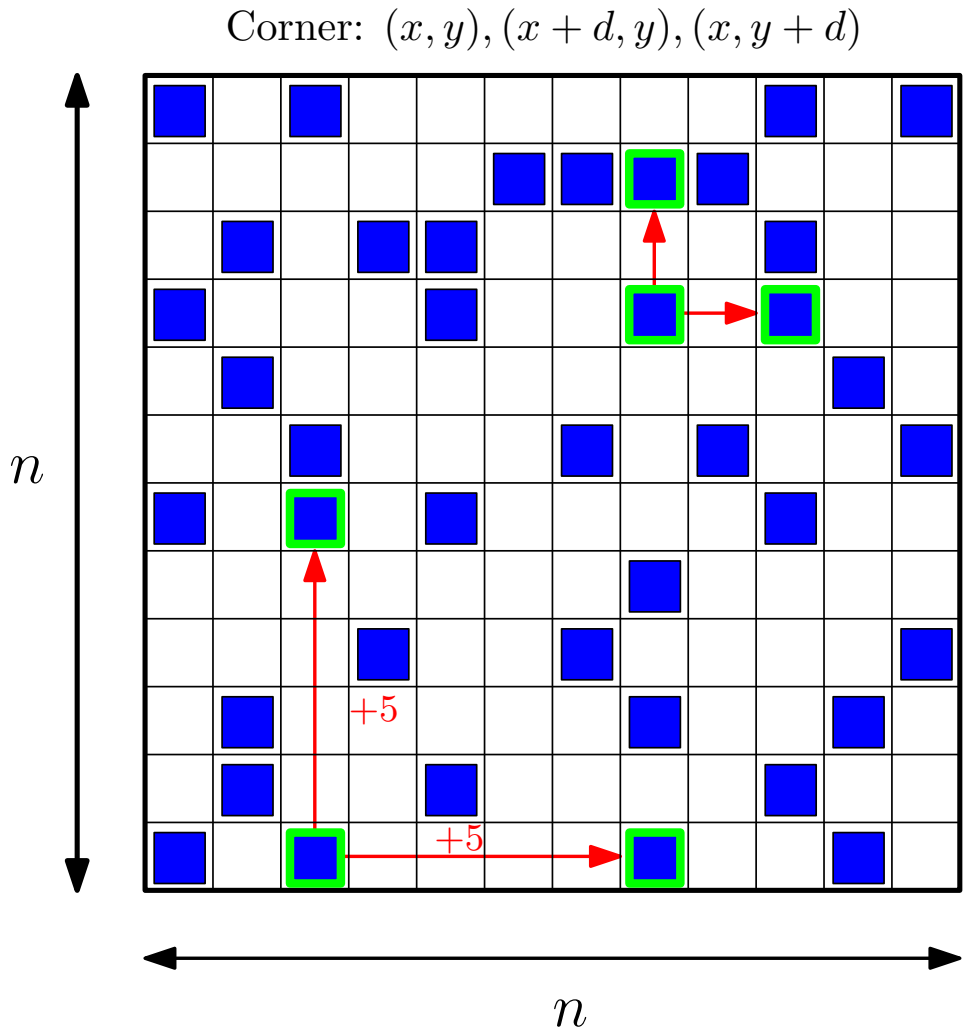
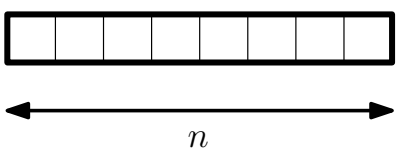


Corners theorem

Corners theorem [Ajtai, Szemerédi, 1974]

$\forall \epsilon > 0$, there is an N such that
Any $S \subset [N]^2$ with density $\geqslant \epsilon$ contains a corner.

Corner theorem $\Rightarrow$ Roth theorem (folklore)



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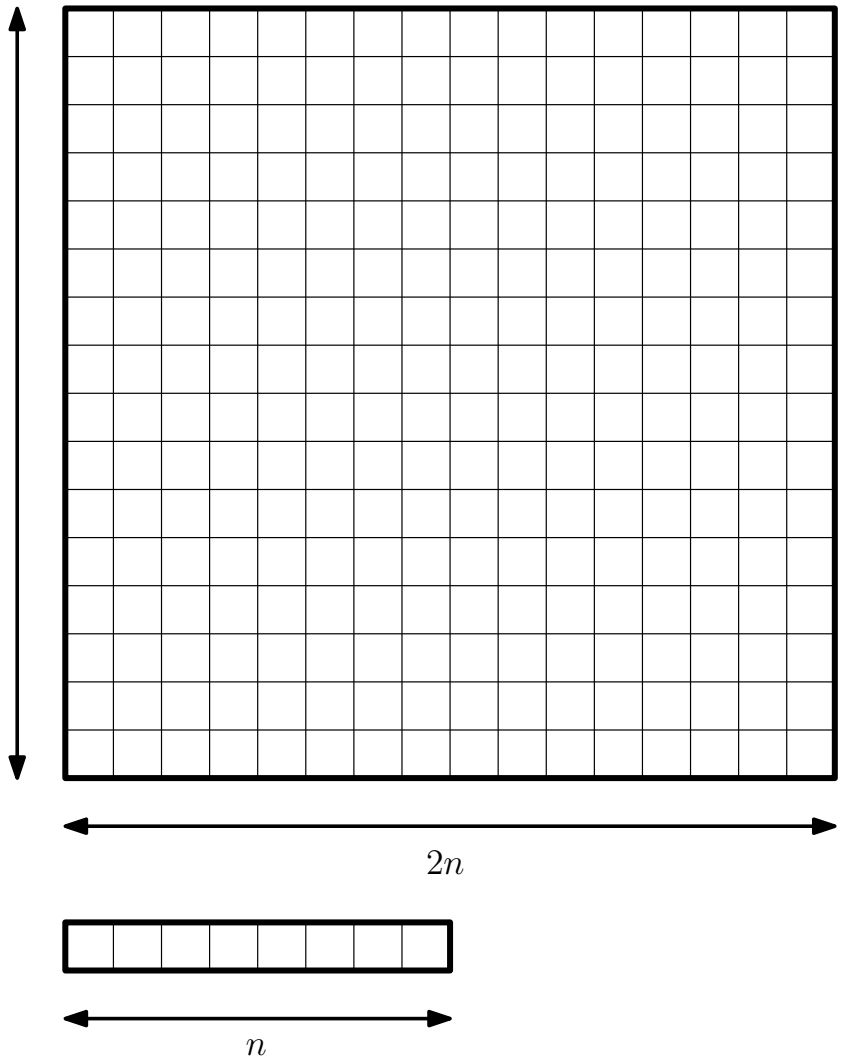

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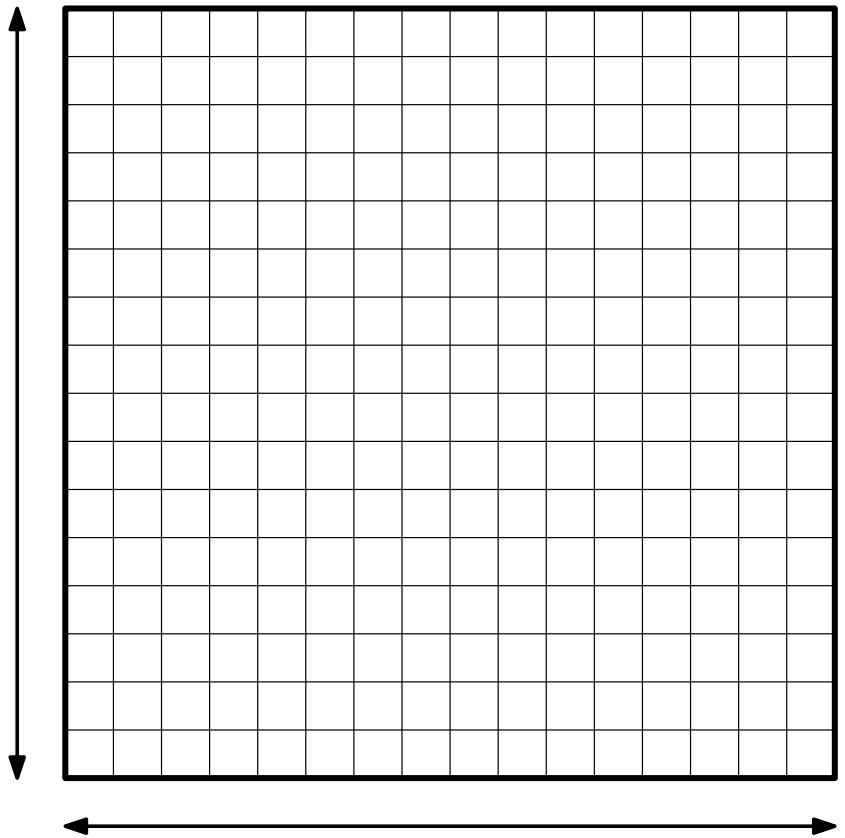

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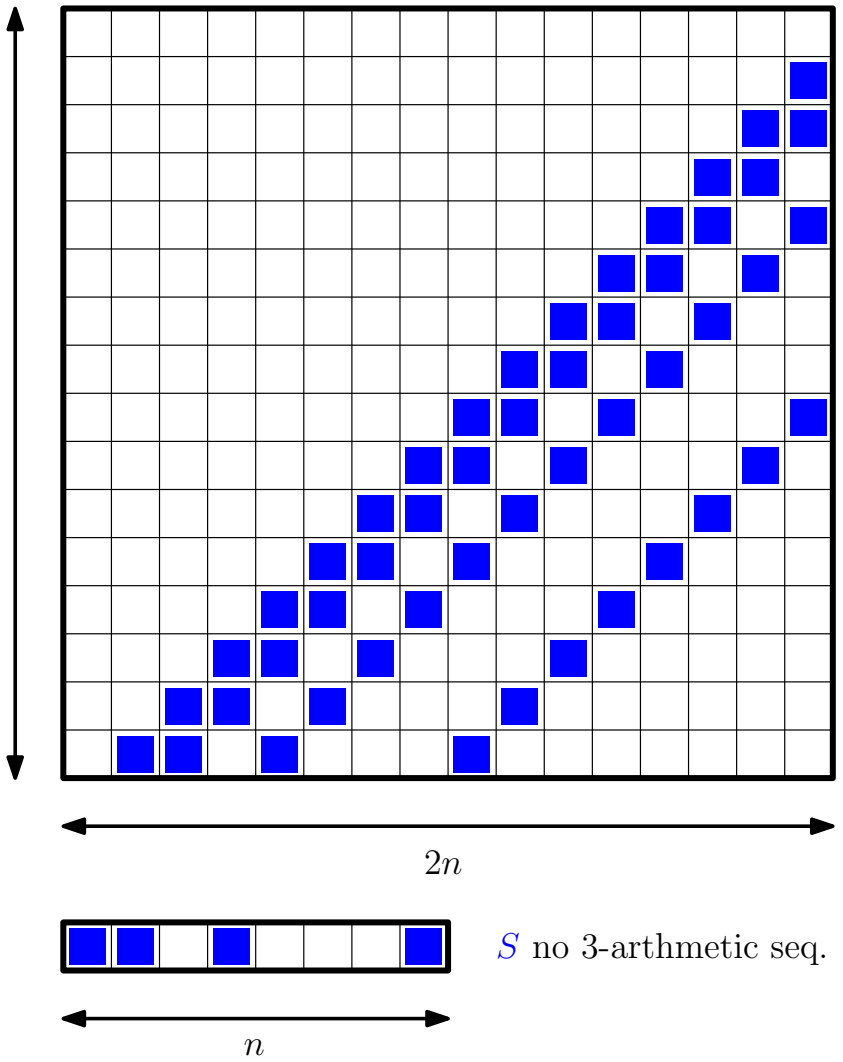

Build A $(x, y) \in A$ iff $x - y \in S$

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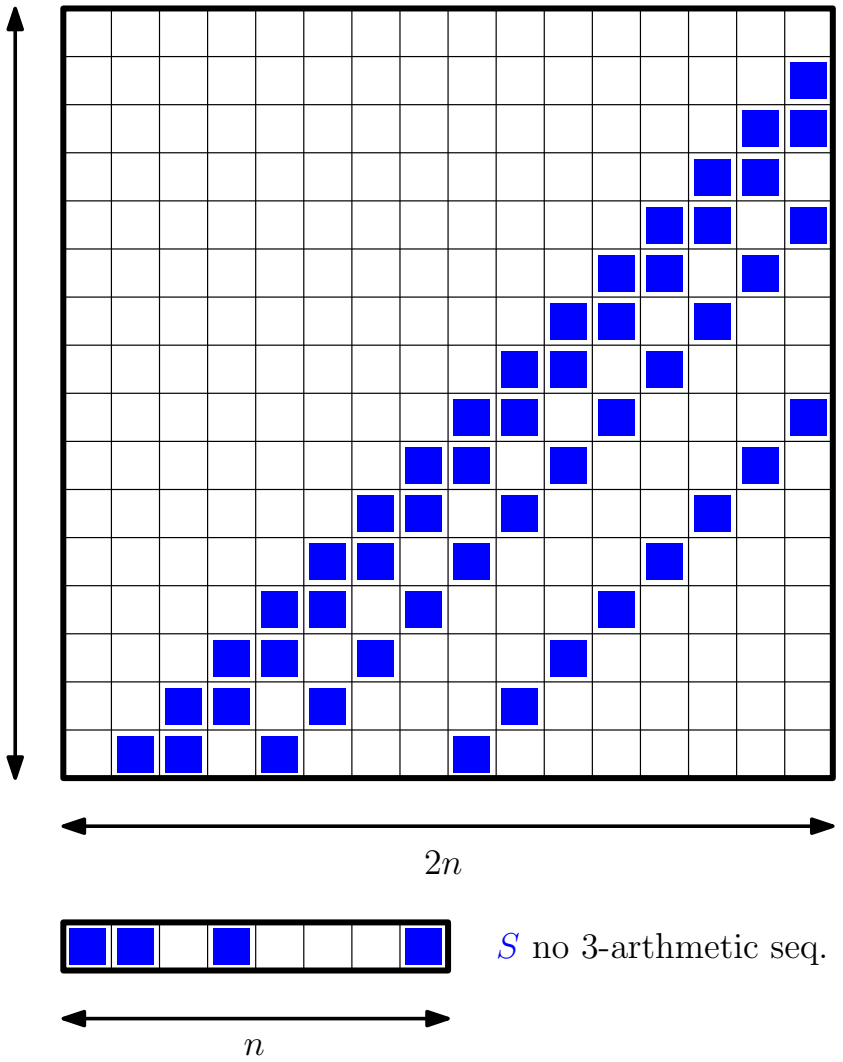

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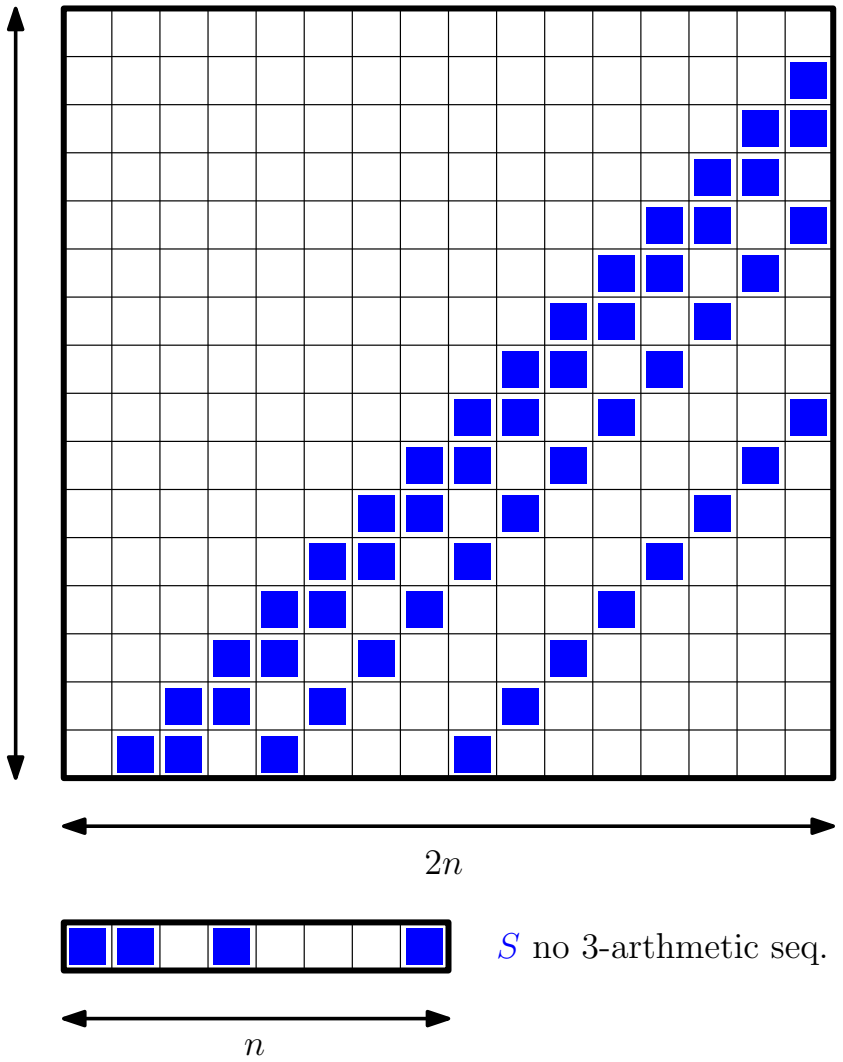

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$(x + d, y), (x, y), (x, y + d) \in A$

$\Rightarrow x - y - d, x - y, x + d - y \in S$

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Szemerédi's theorem

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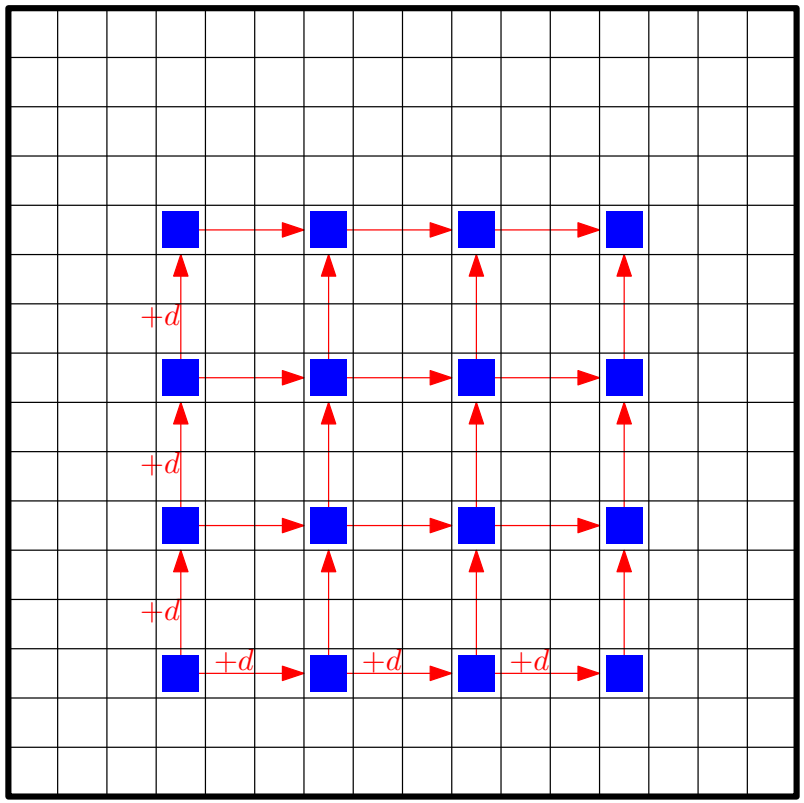
Szemerédi's theorem, 1975

$\forall \epsilon > 0$, $\forall k \in \mathbb{N}^+$, there is a (very large) $N := N(\epsilon, k)$ such that:
$\forall S \subseteq [N]$, density$(S) \geqslant \epsilon \Rightarrow S$ contains a k-arithmetic sequence.

Multi-dimensional Szemerédi's theorem

The k a-dimensional grid

$$\{\overline{x} + (k_1 d, k_2 d, \ldots, k_a d) \ : \ k_i \in [k]\}$$



Multi-dimensional Szemerédi's theorem

Multidim theorem [Furstenberg, Katznelson, 1985]

$\forall \epsilon > 0$, $\forall k \in \mathbb{N}^+$, there is a (very large) $N := N(\epsilon, k) \in \mathbb{N}$ such that $\forall S \subseteq [N]^d$, density$(S) \geqslant \epsilon \Rightarrow S$ contains a k d-Grid.

The question

Szemerédi's theorem

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Question

When $N \to \infty$, how small needs $\epsilon := \epsilon(N)$ to be to avoid:

- 3-arithmetic progressions?
- Corners?
- General d-dimensional grids?

What is known about corners?

Theorem [Behrend, 1946]

There is a corner-free set of density $2^{-2.879\sqrt{\log N}}$.

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Theorem [Shkredov, 2006]

There is no corner-free set of density $1/(\log\log N)^c$.

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Best upper bound [Jaber, Liu, Lovett, Ostuni, Sawhney, 2025]

There is no corner-free set of density $2^{-(\log N)^{1/600}}$.

Triangle removal lemma $\Rightarrow$ corners theorem

Triangle removal lemma [Ruzsa, Szemerédi 1978]

For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that:
If one cannot erase **all** triangles of G by deleting ϵn^2 edges,
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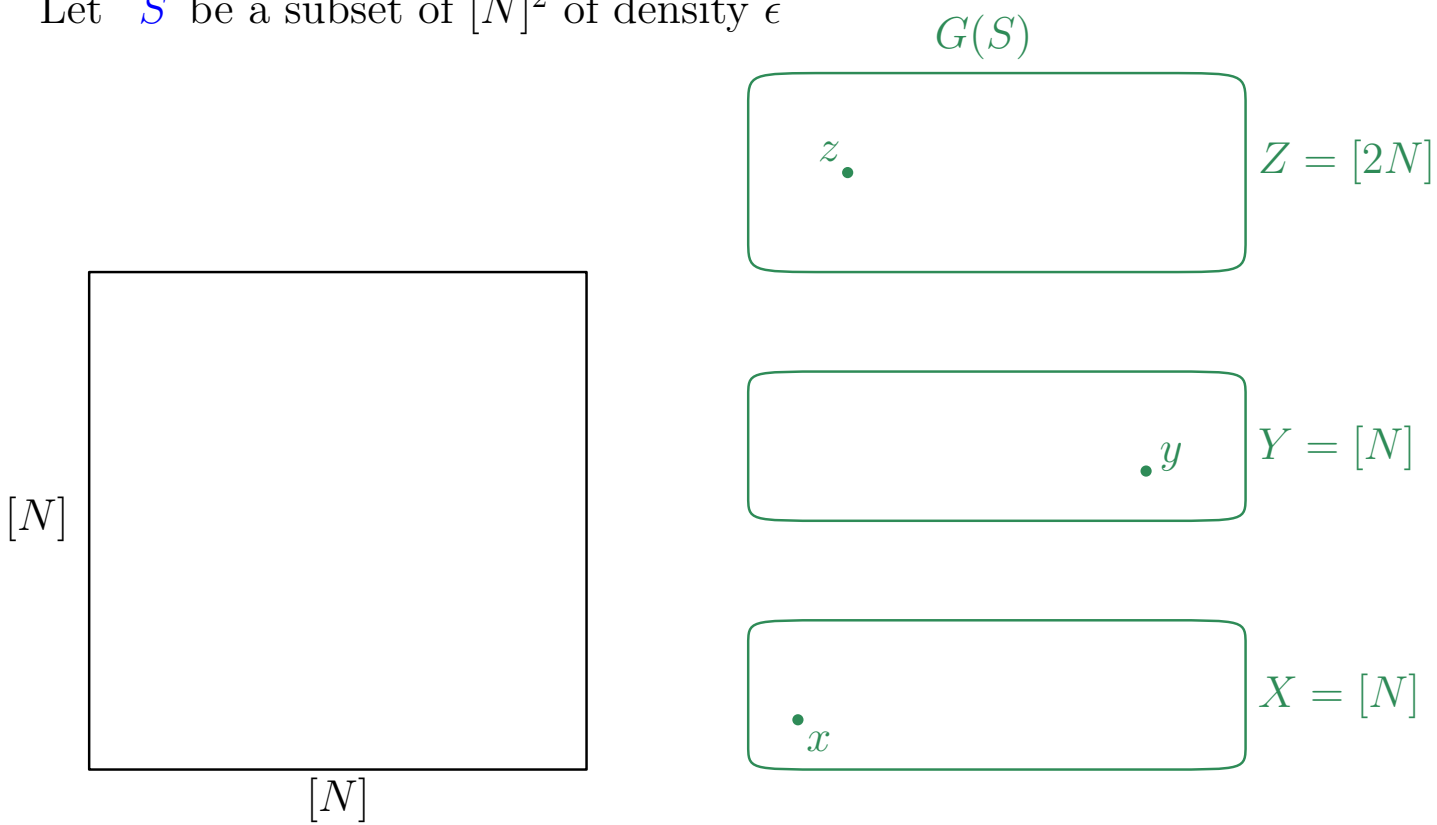

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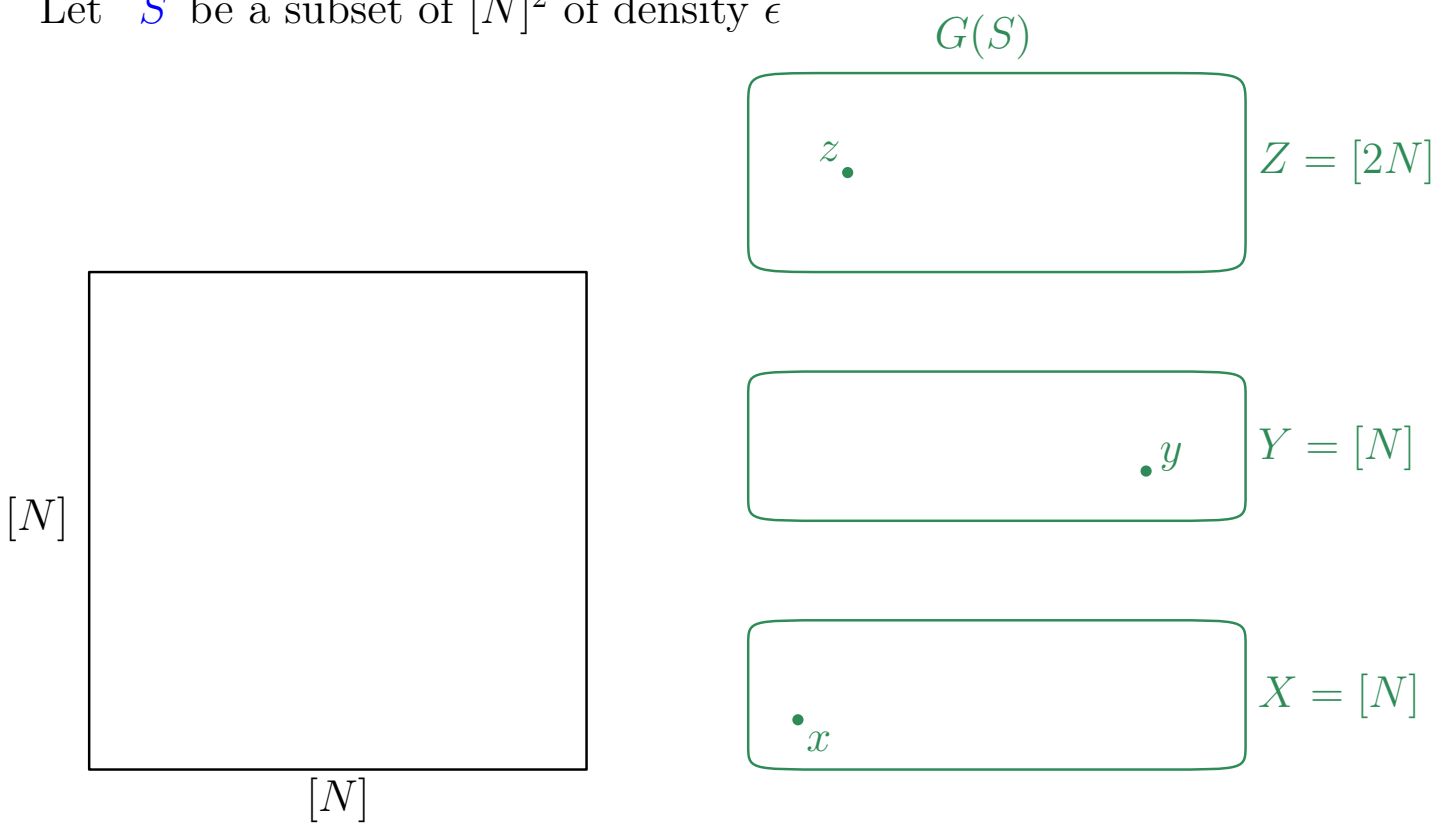

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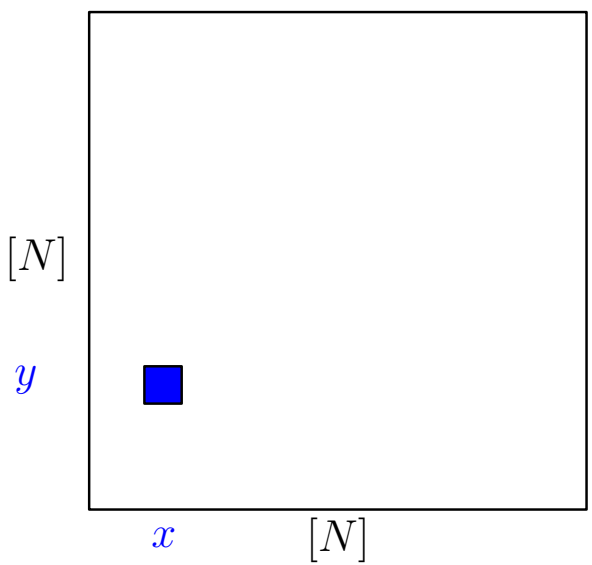

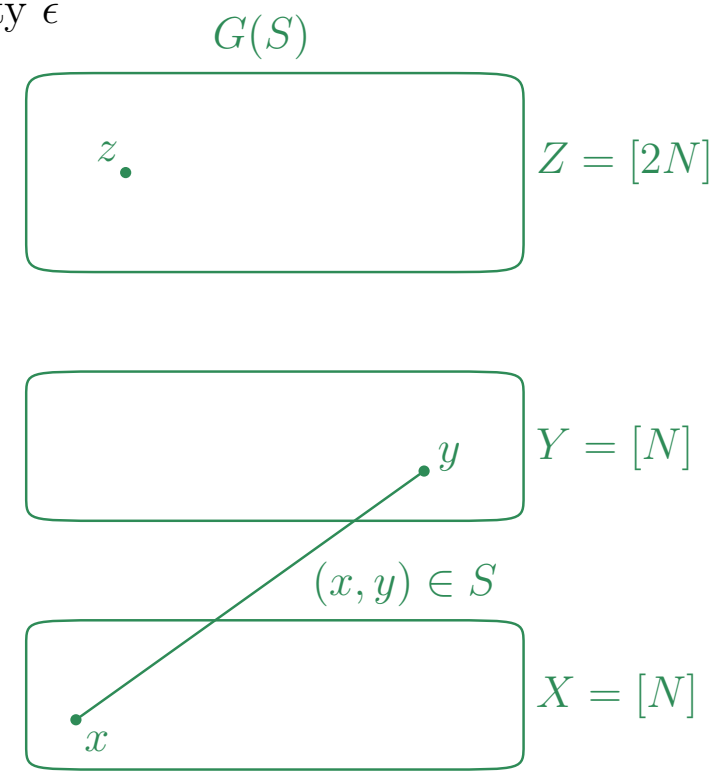

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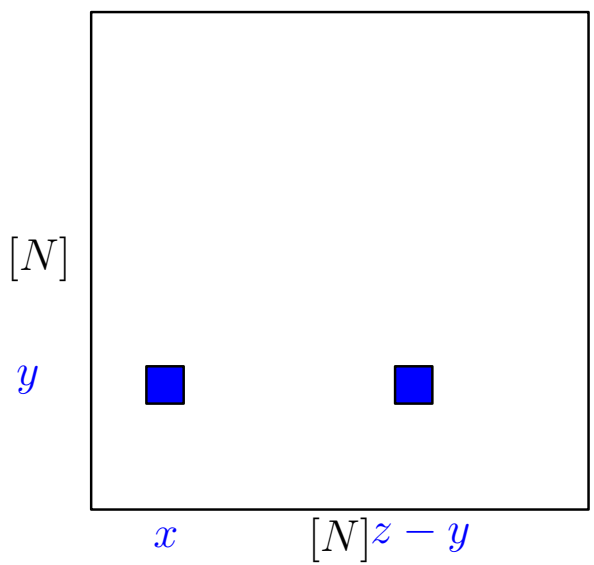

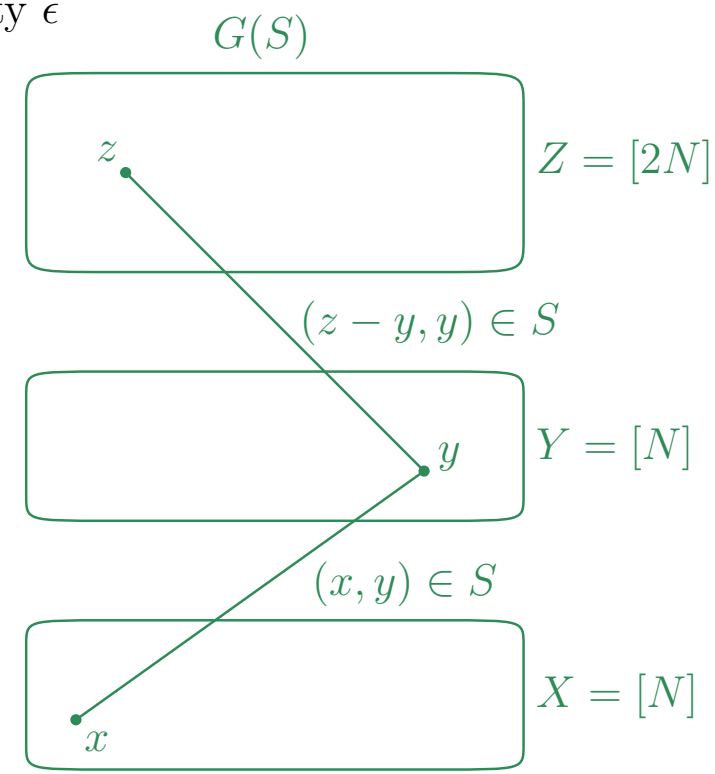

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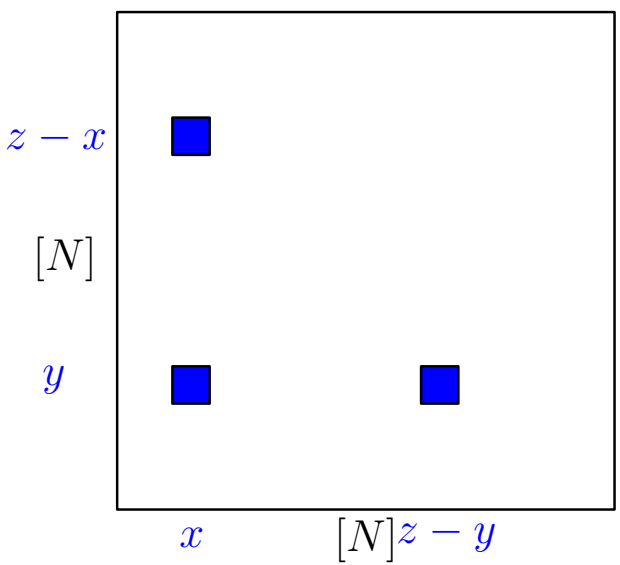

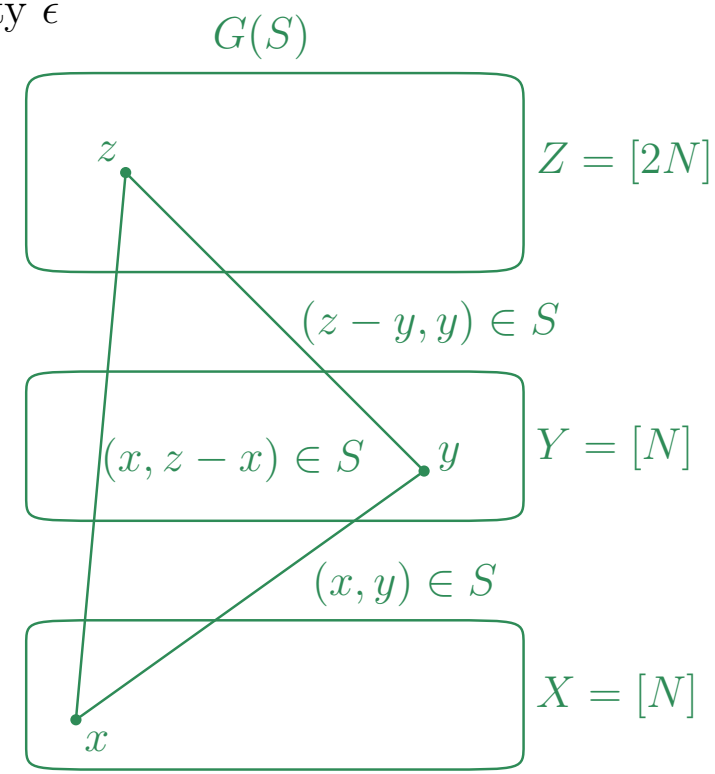

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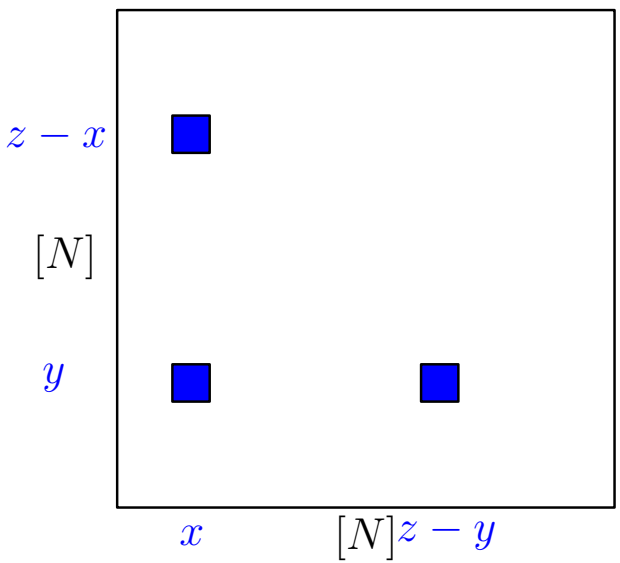
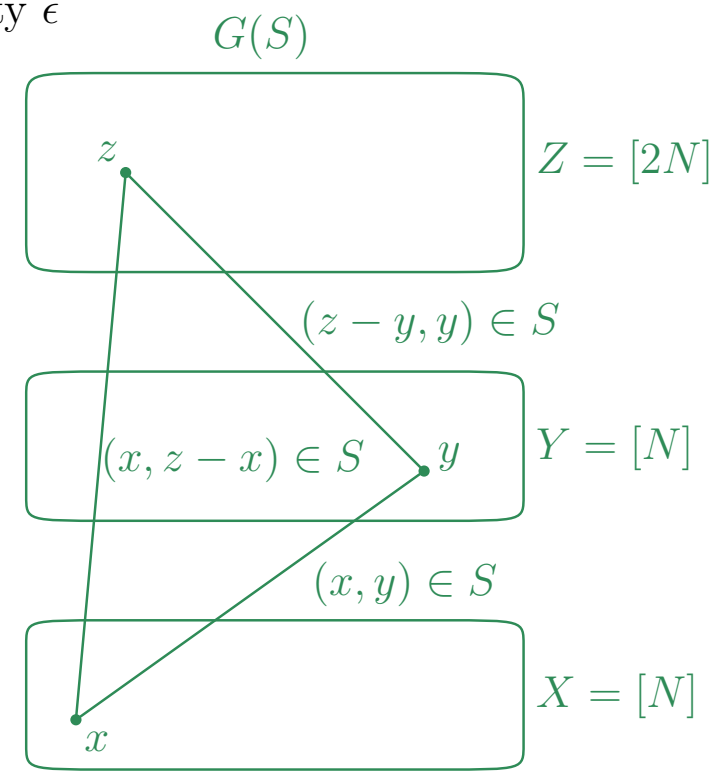
Triangle $\Leftrightarrow$ (possibly trivial) corner





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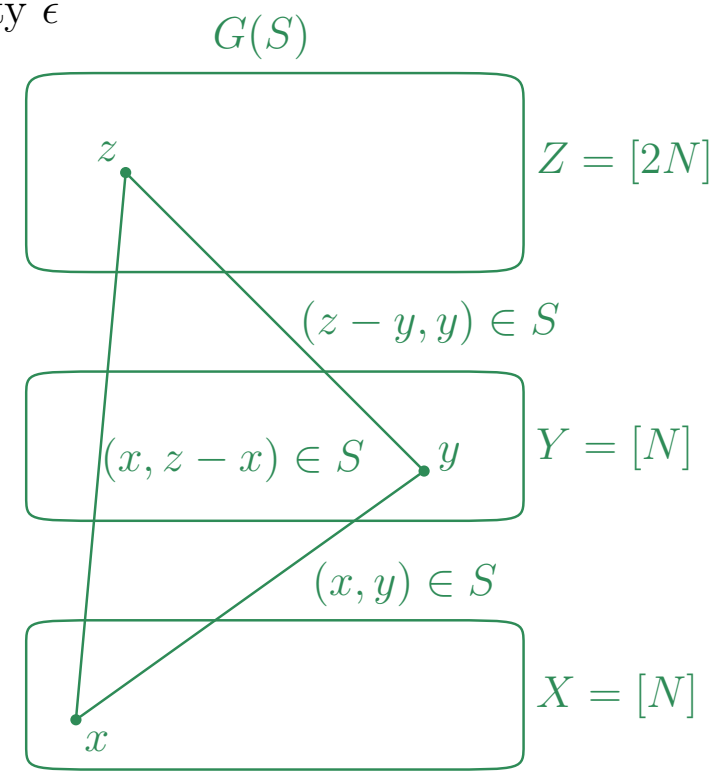

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Trivial corner $(x, y), (x, y), (x, y)$
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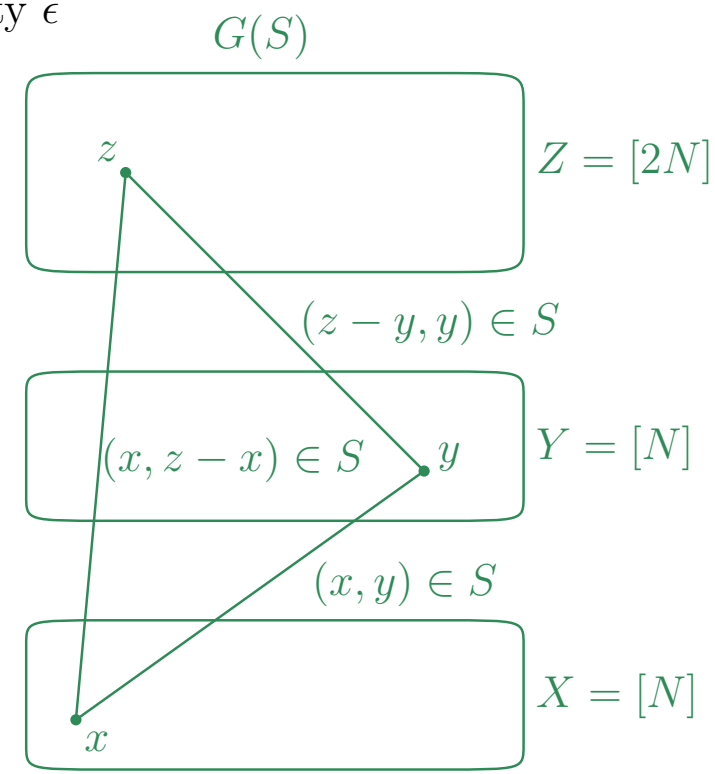

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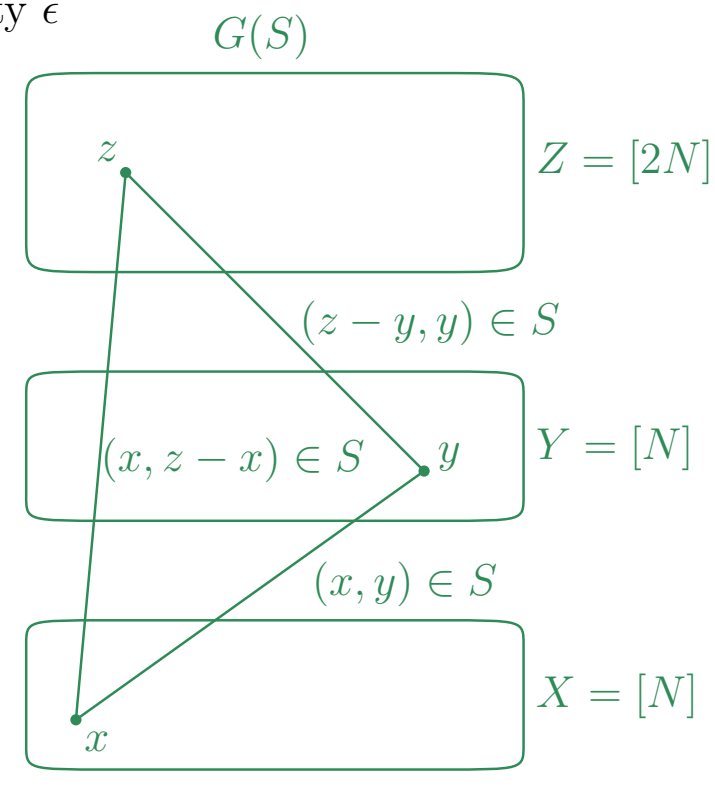

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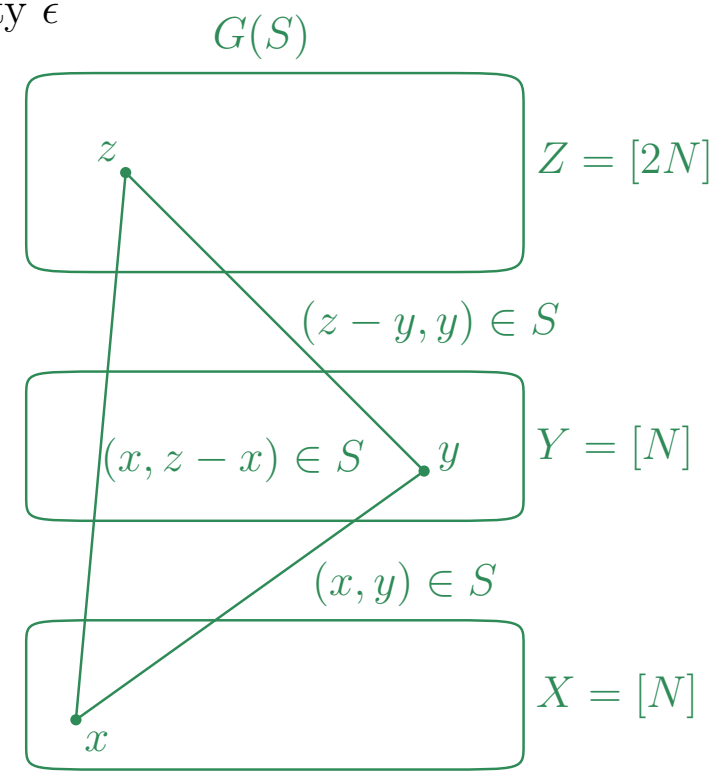

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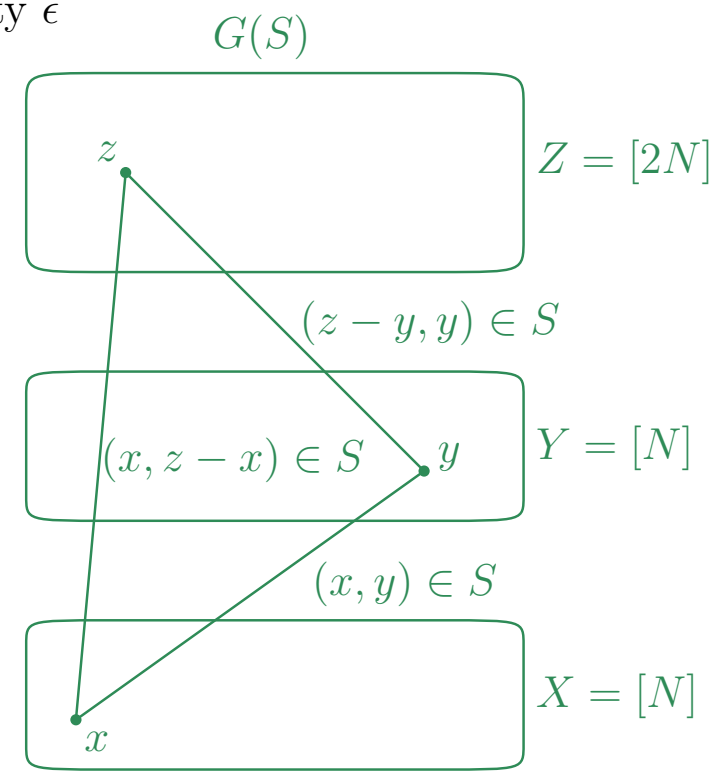
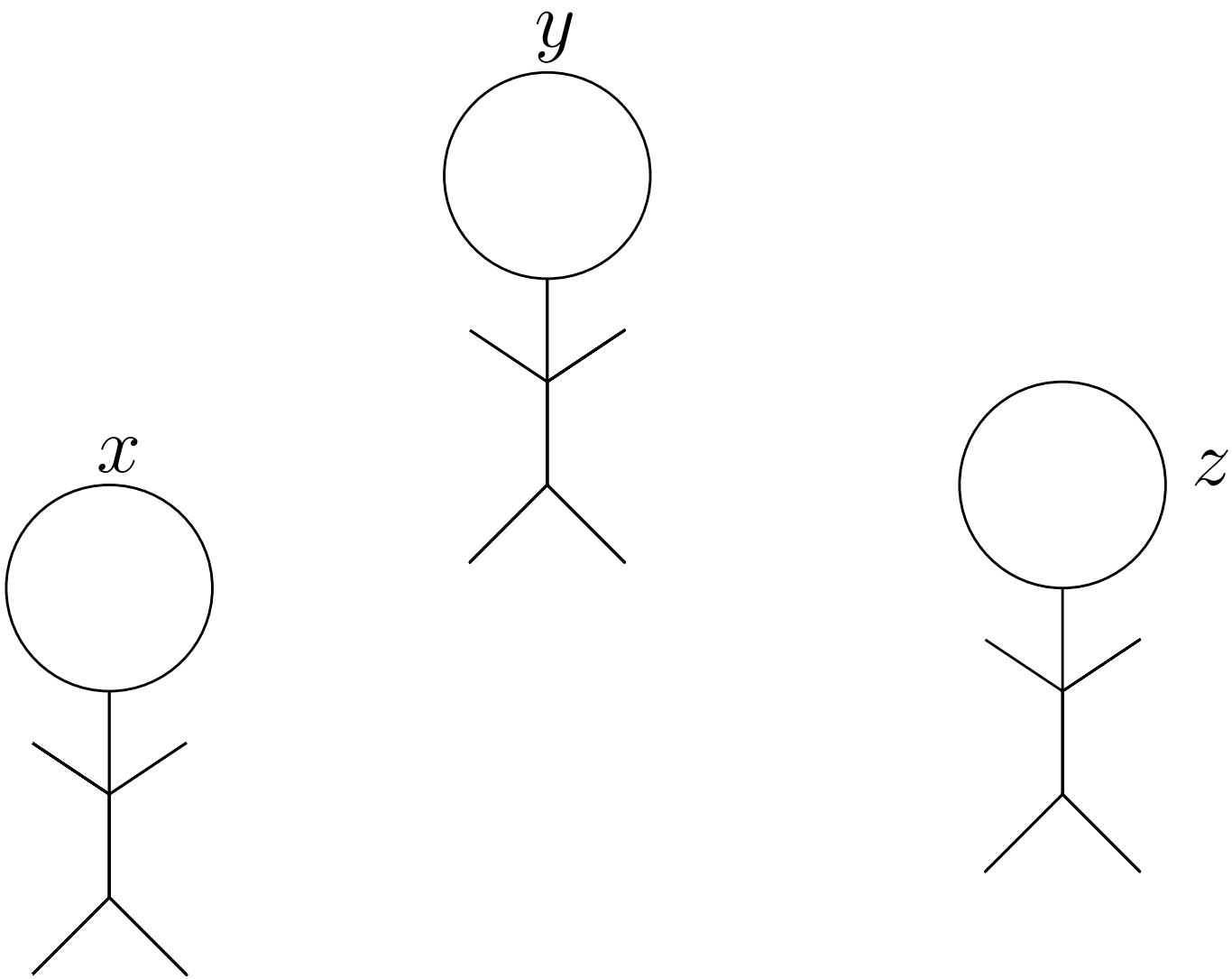
$\rightarrow \delta N^3$ triangles

$\rightarrow \delta N^3 - N^2$ non-trivial corners



The 3-NOF protocol



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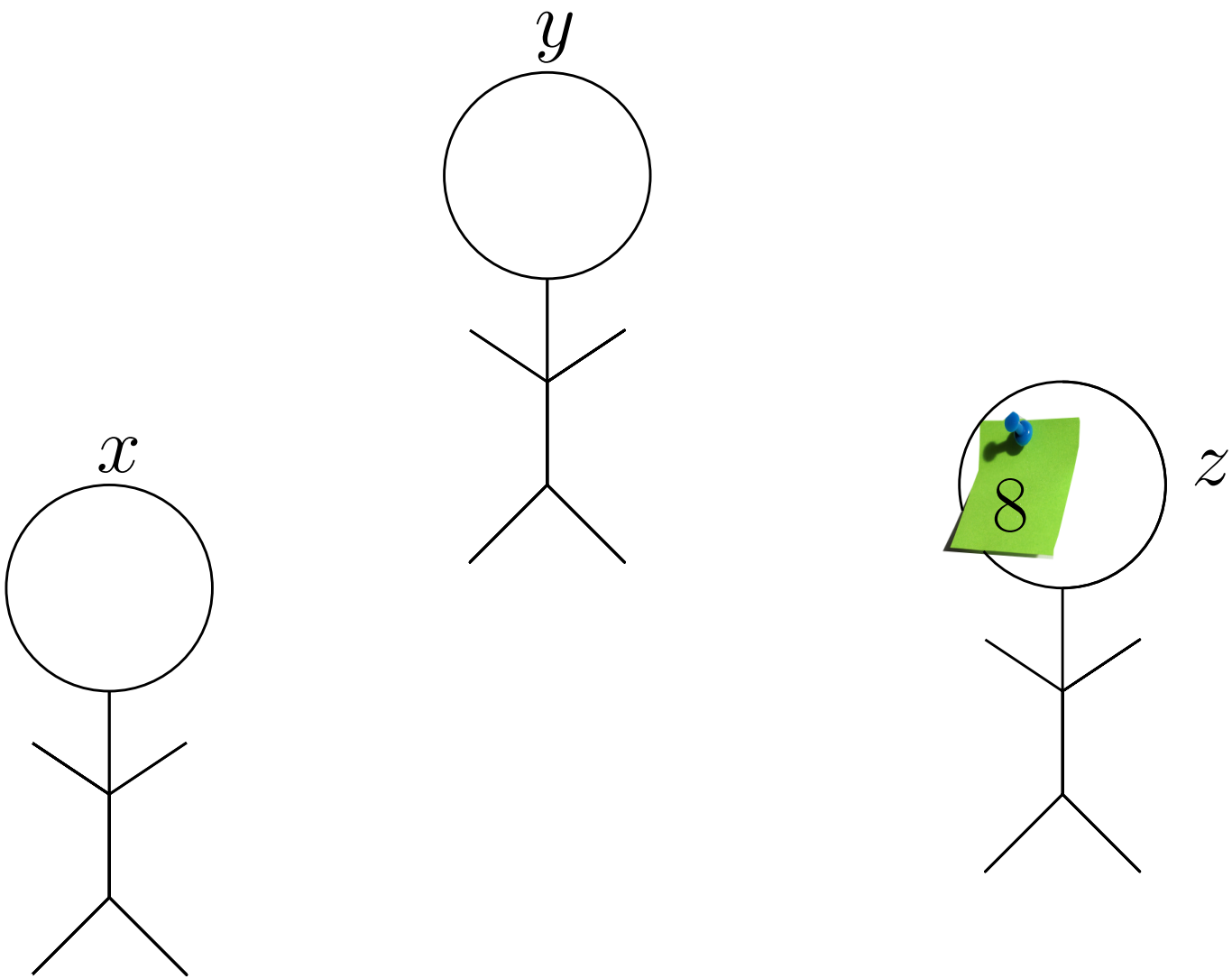

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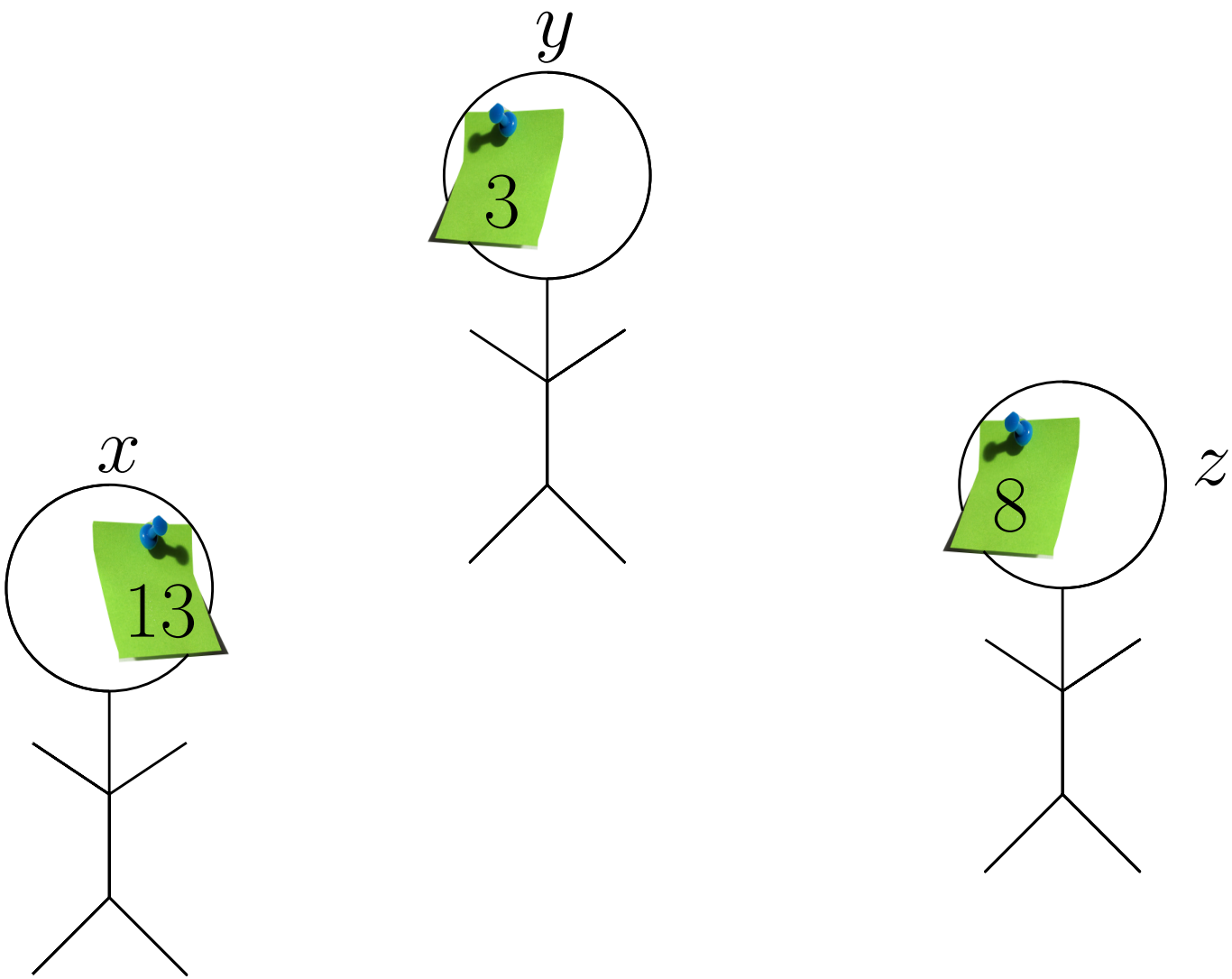

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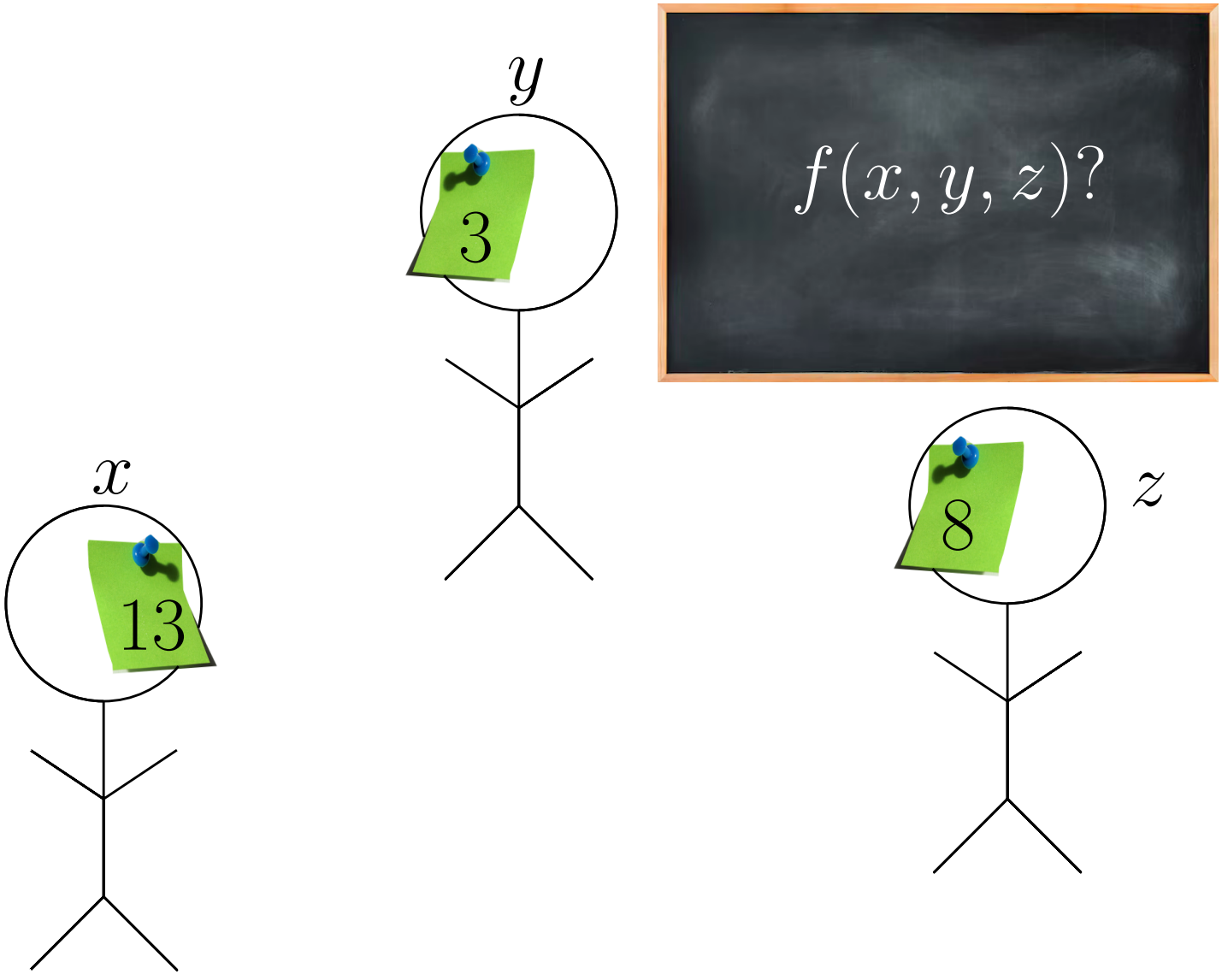

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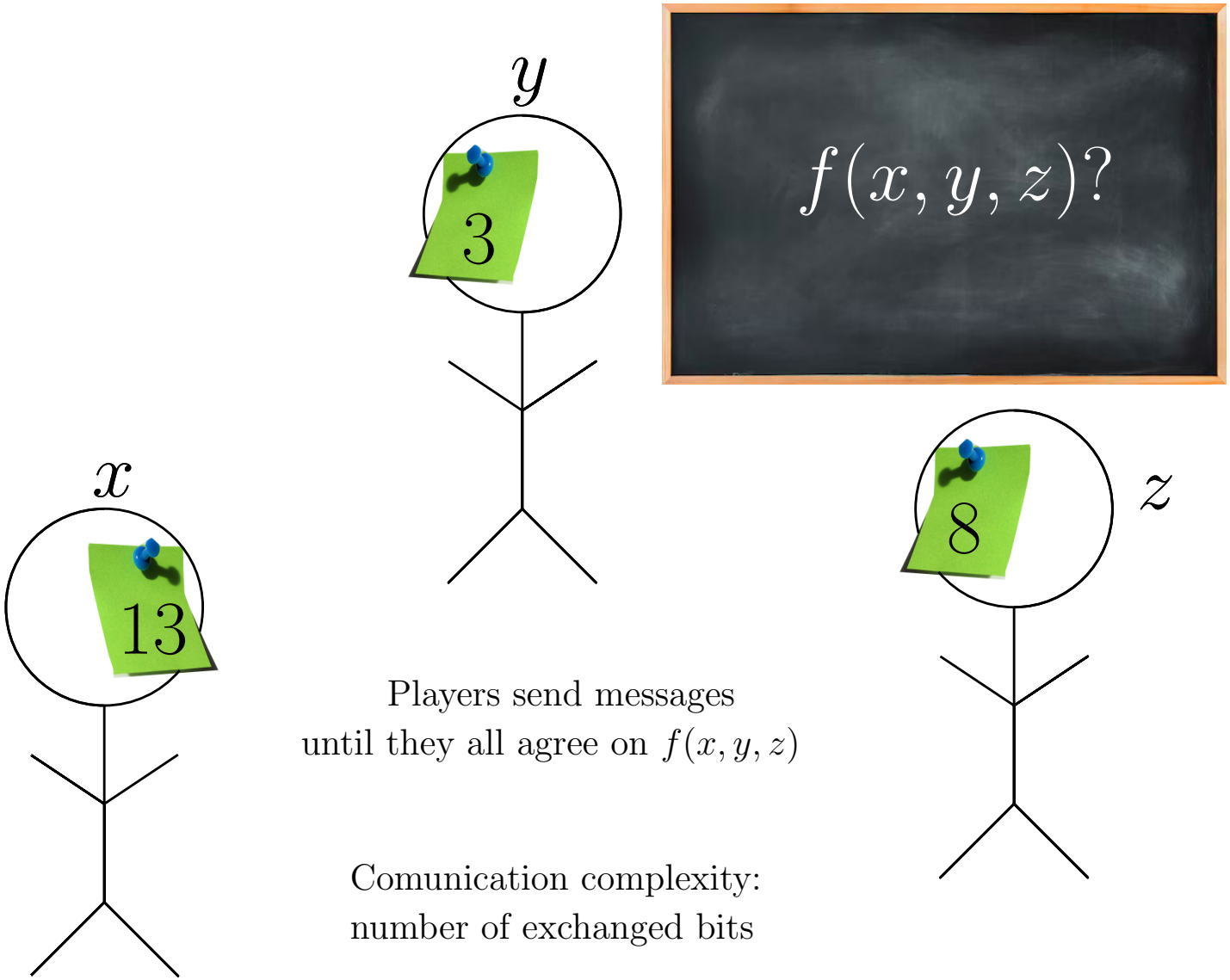
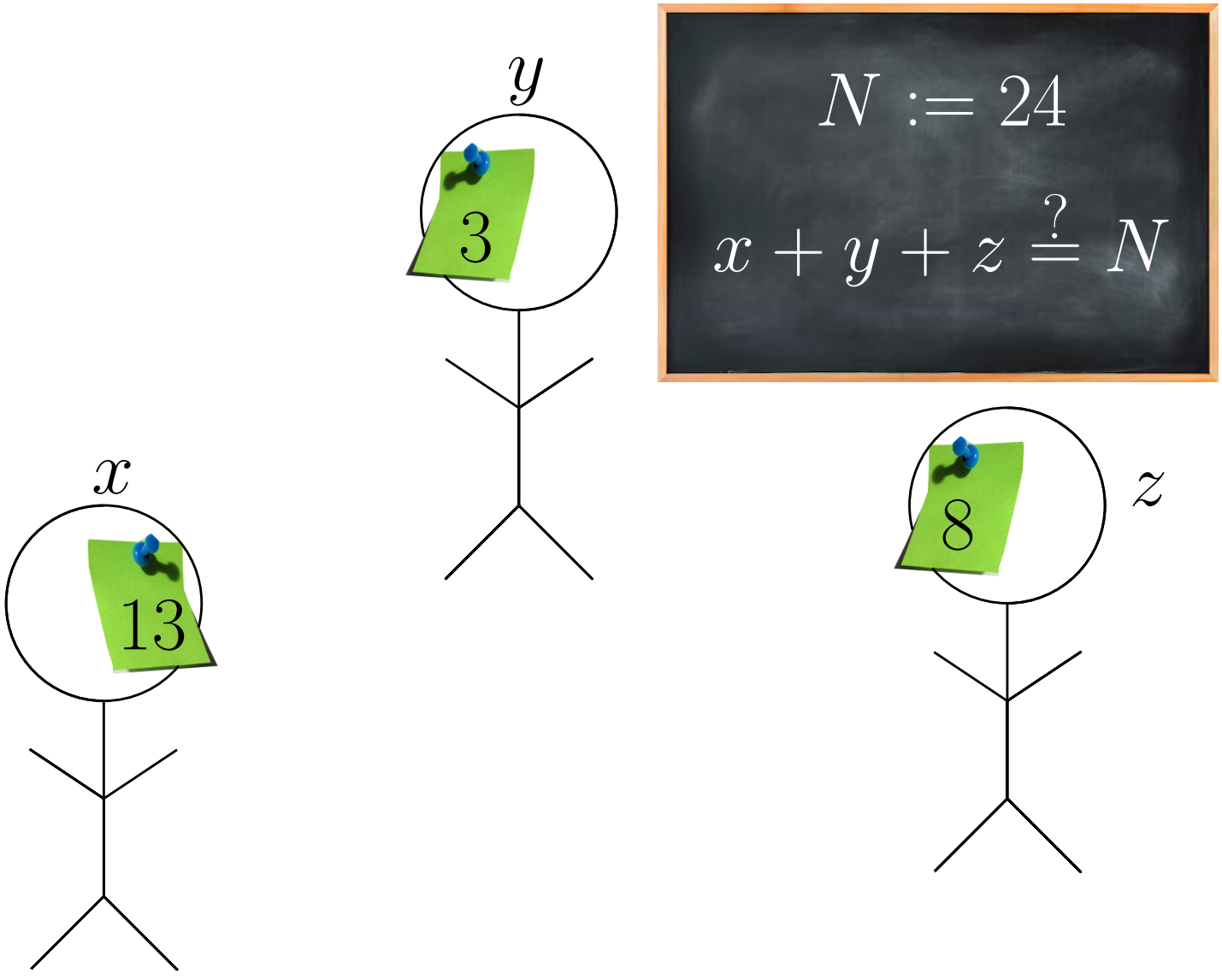


Players send messages
until they all agree on $f(x, y, z)$

Comunication complexity:
number of exchanged bits

One round 3-NOF exactly-*N* protocol



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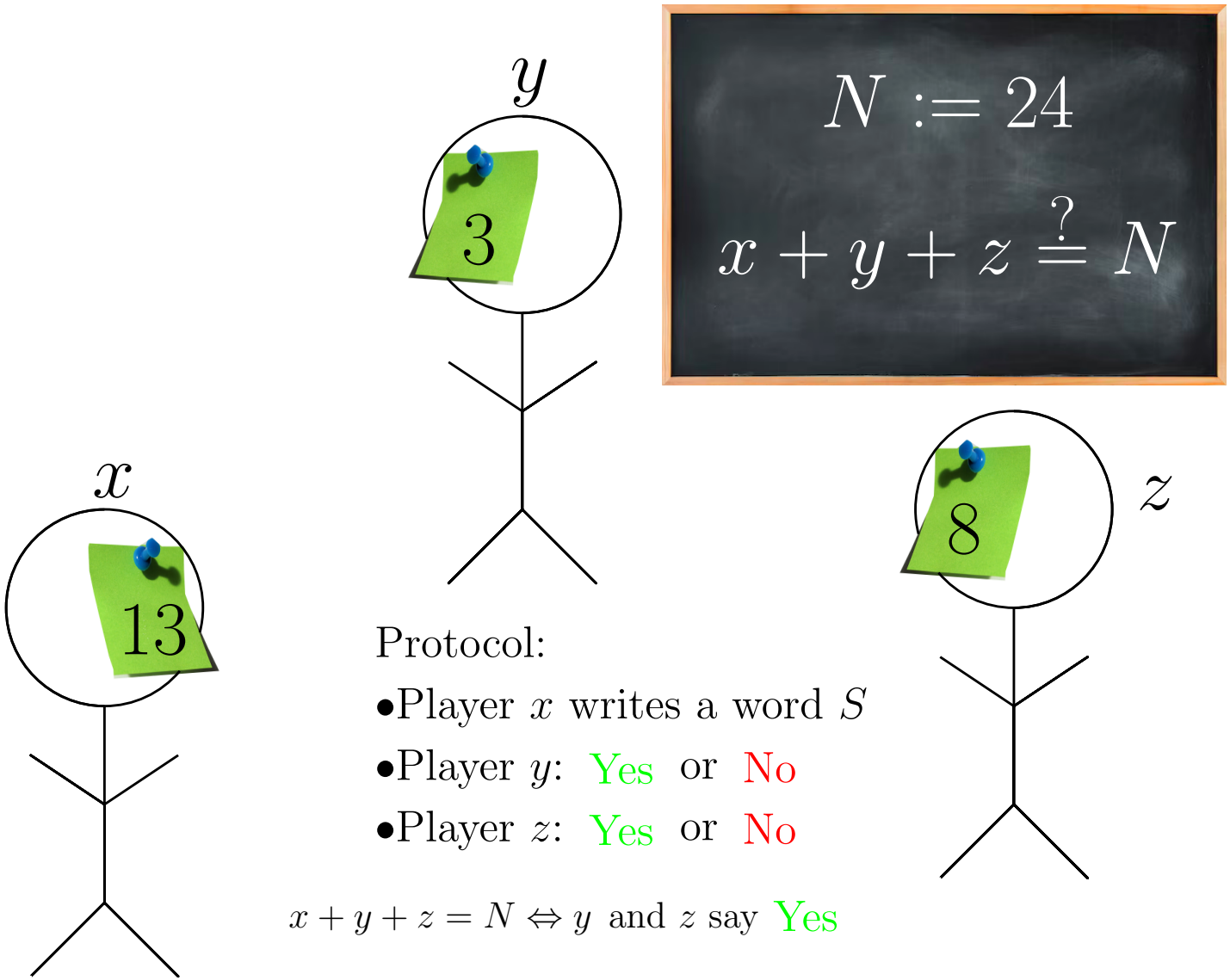

Protocol:

- Player x writes a word S
- Player y: Yes or No
- Player z: Yes or No

$x + y + z = N \Leftrightarrow y$ and z say Yes

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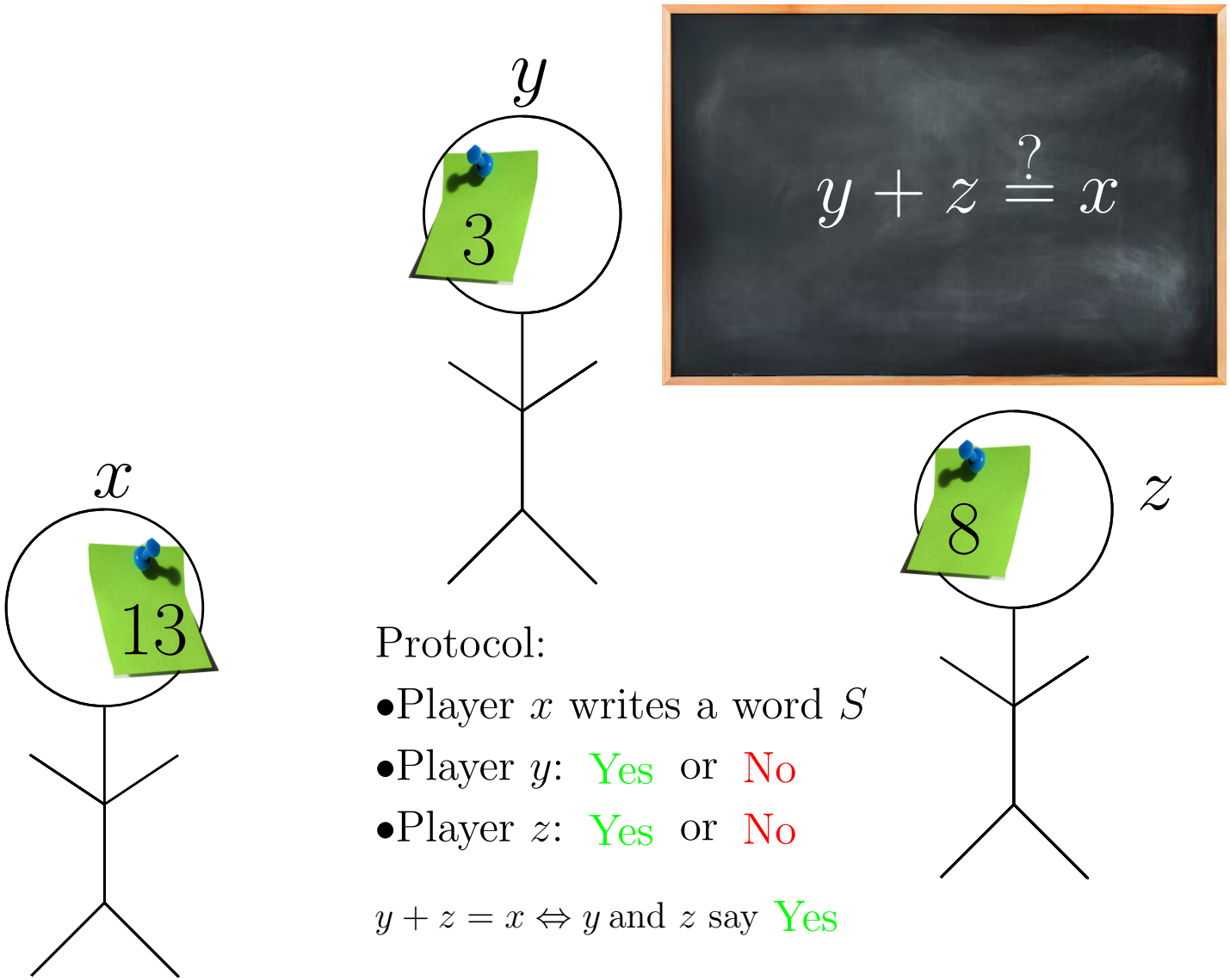

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Good protocol $\Rightarrow$ big corner-free set

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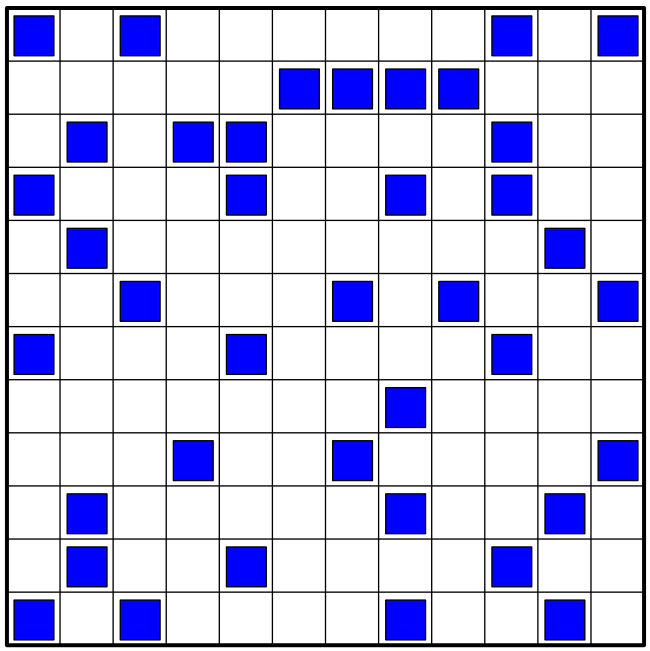

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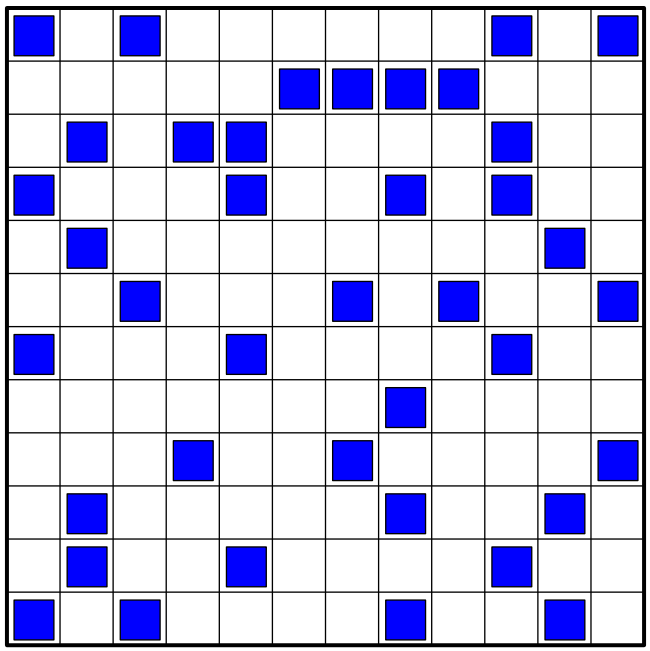

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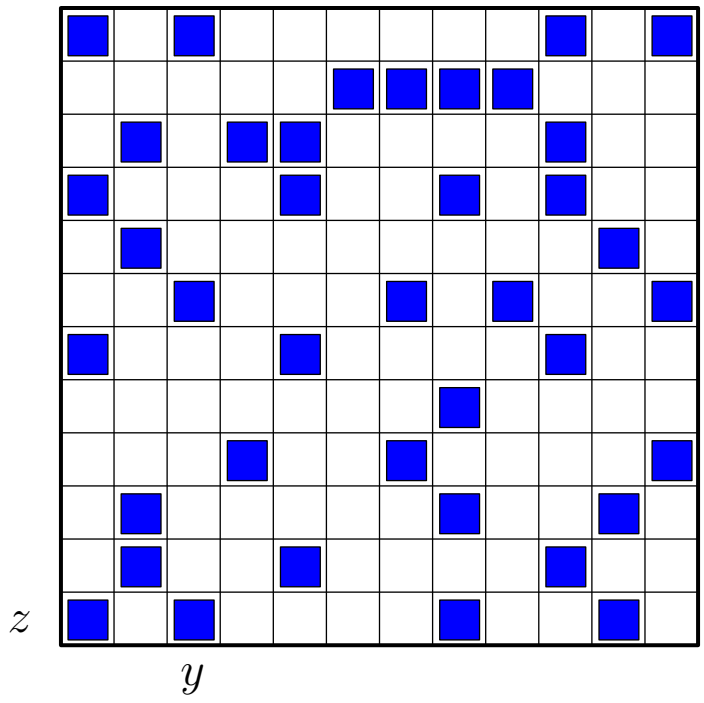

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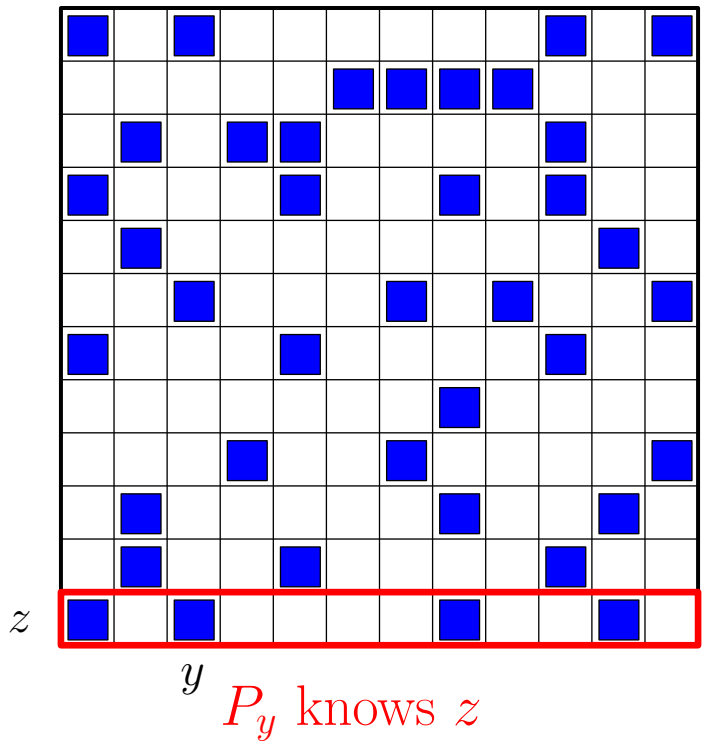

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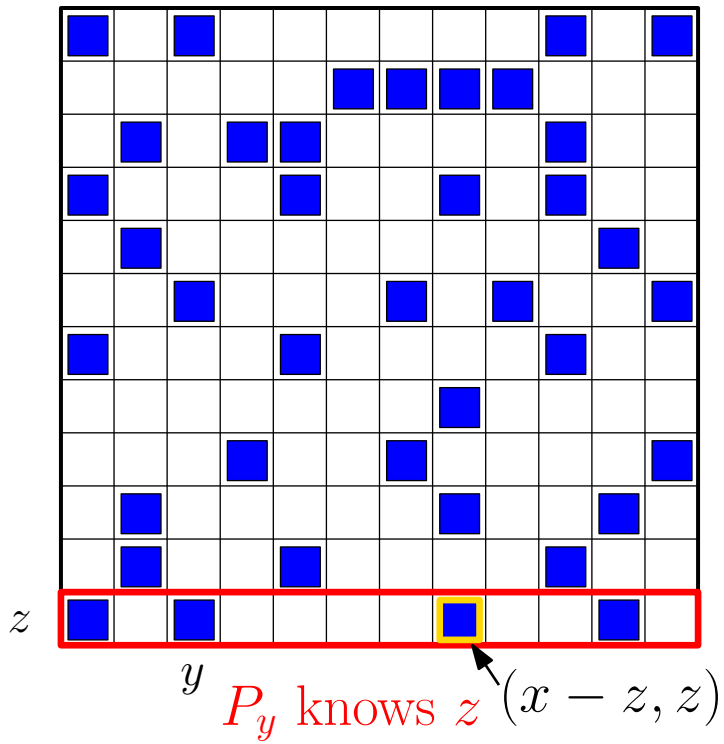

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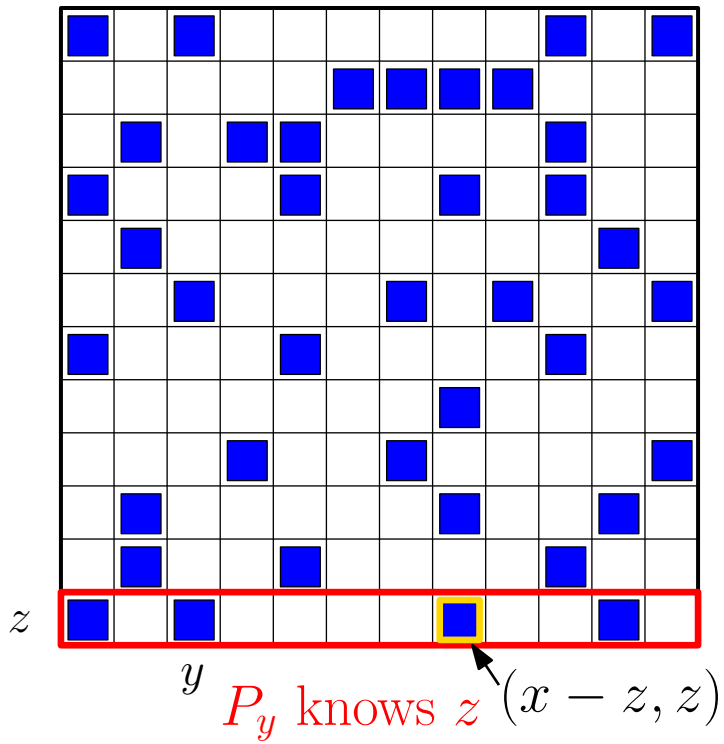

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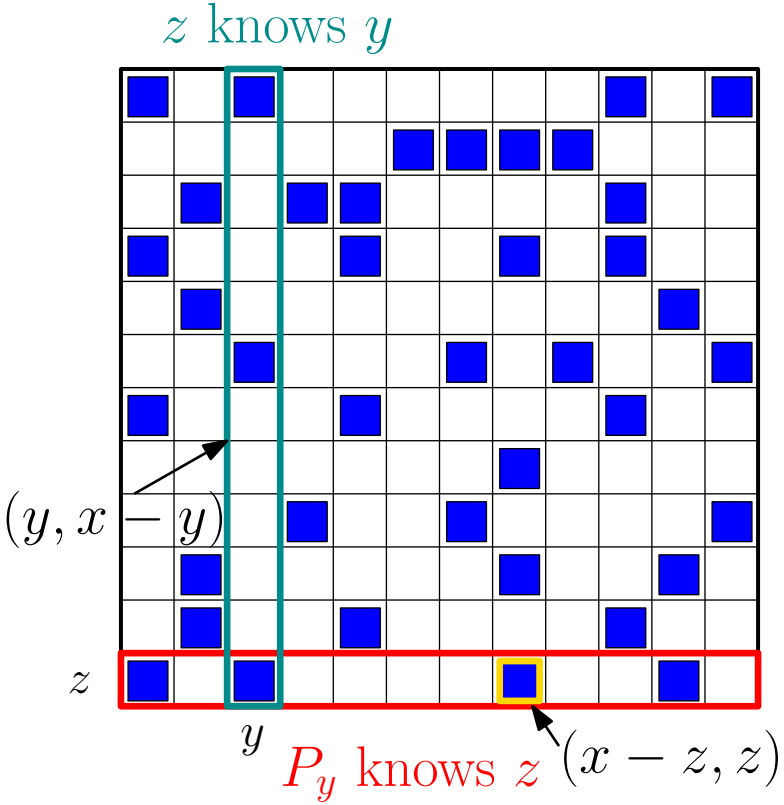

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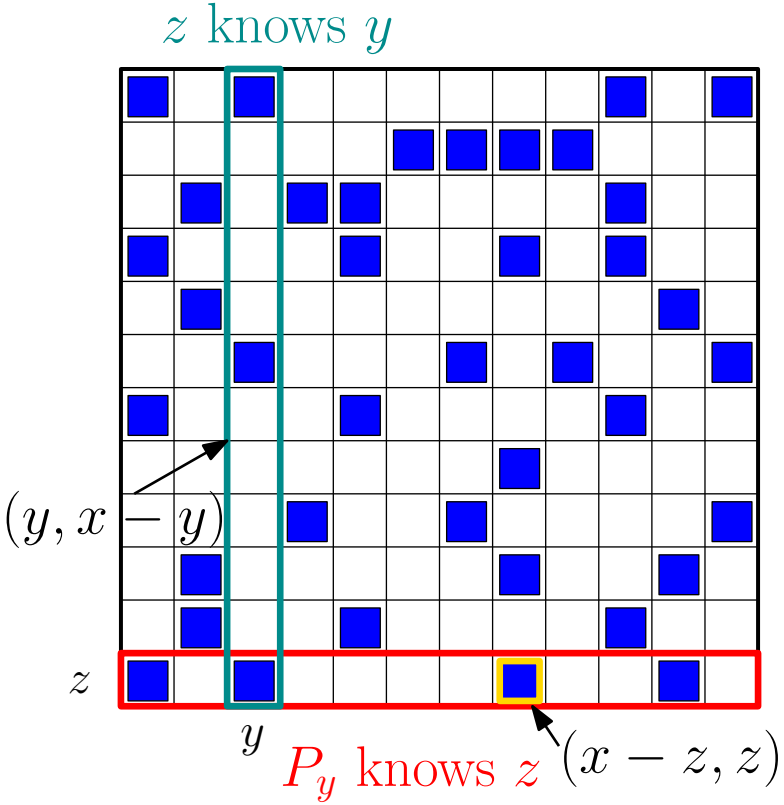

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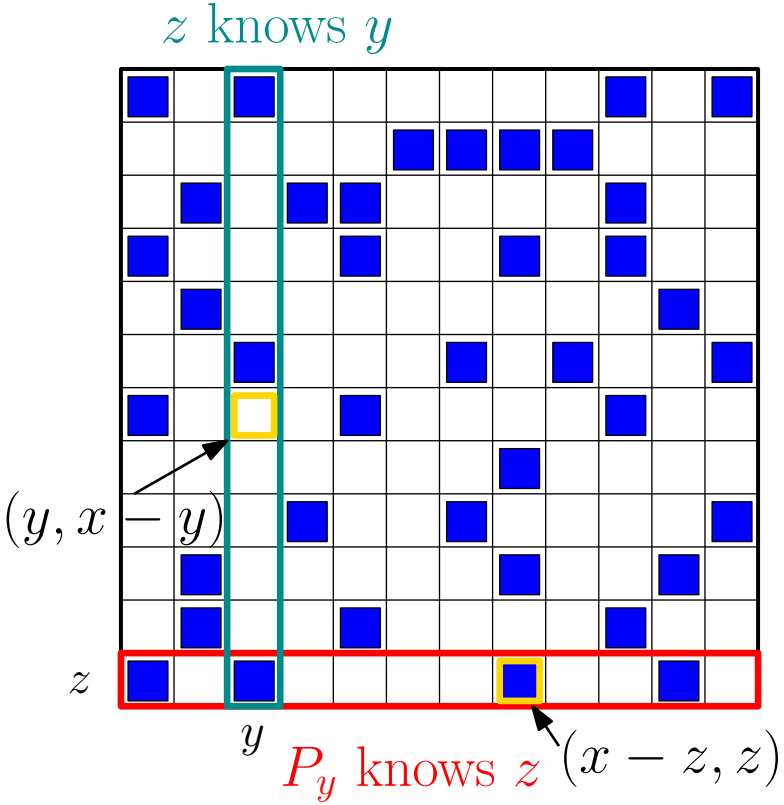

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$\Rightarrow \forall d, \forall (y,z) \in S_w \neg(y+d, z) \in S_w$
or $\neg(y, z+d) \in S_w$

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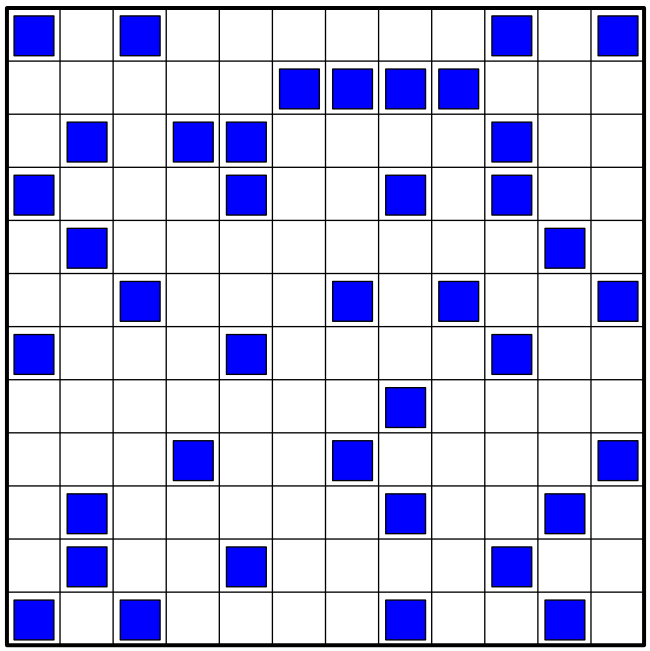

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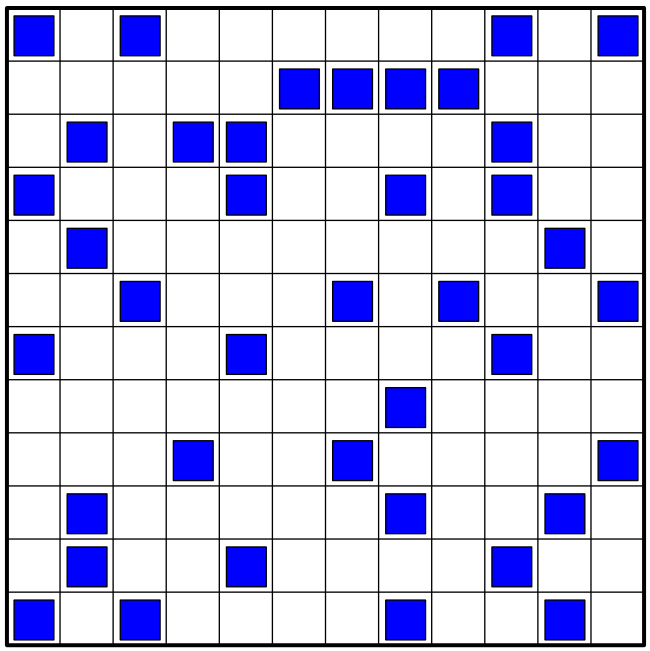

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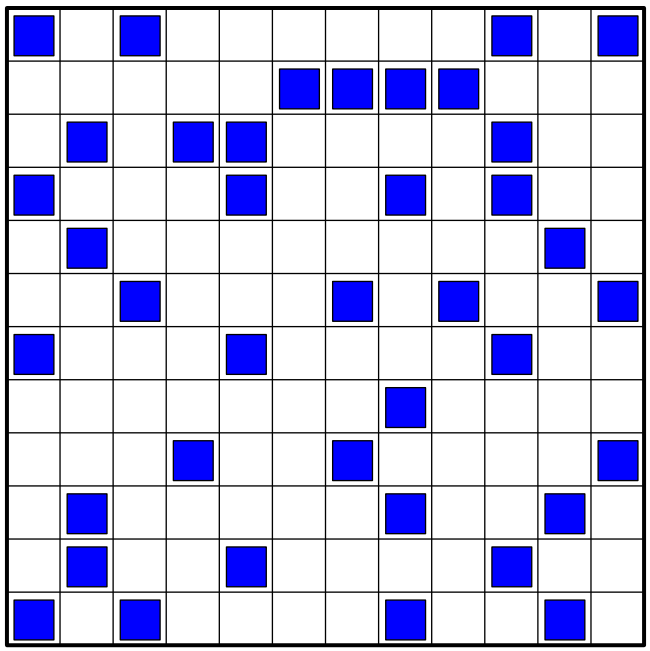

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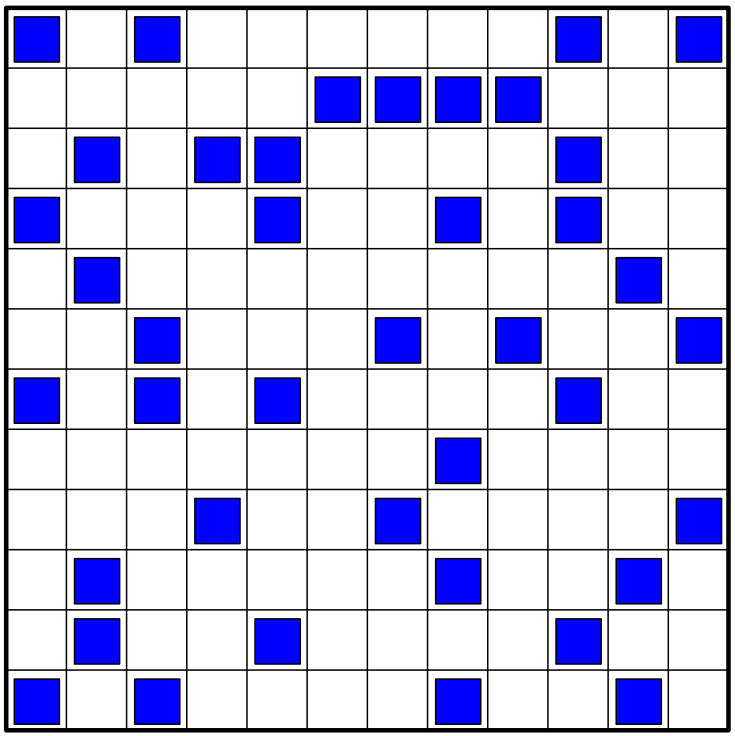
$\exists w,\ \text{density}(S_w) \geq 1/2^{|w|}$

Exists large corner-free sets!

Big corner-free set $\Rightarrow$ good protocol

Big corner-free set $\Rightarrow$ good protocol

Let S be a corner-free set of size k



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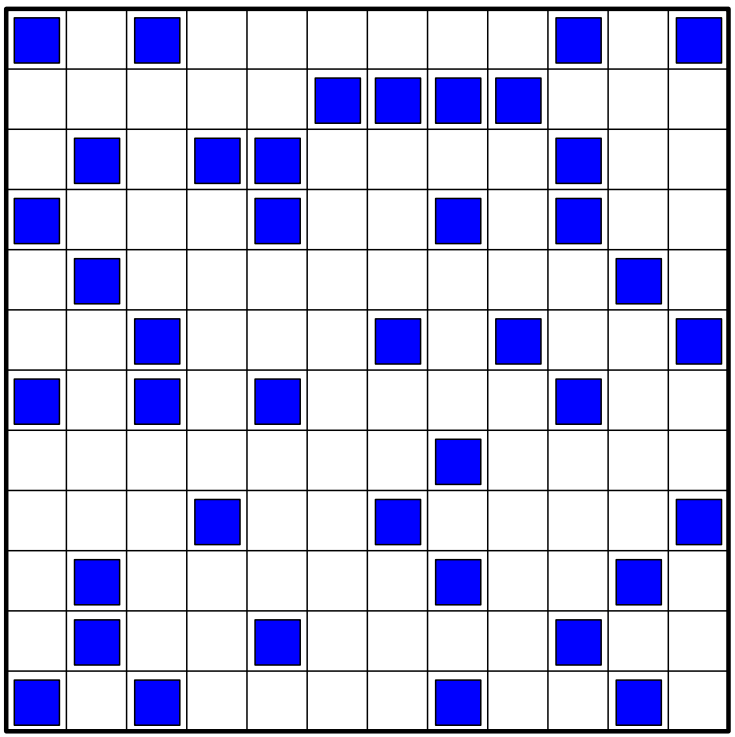

Protocol *under* $(y, z) \in S$

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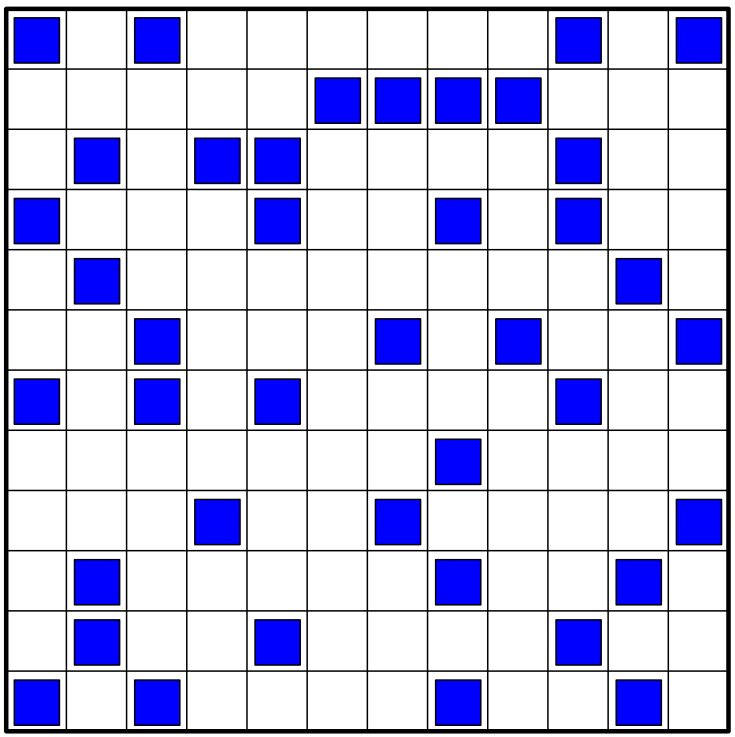

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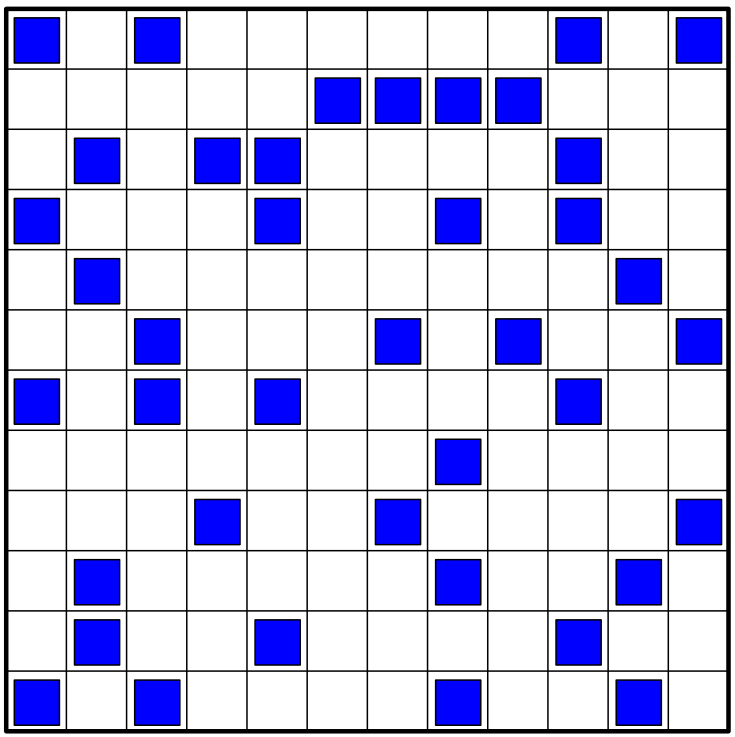

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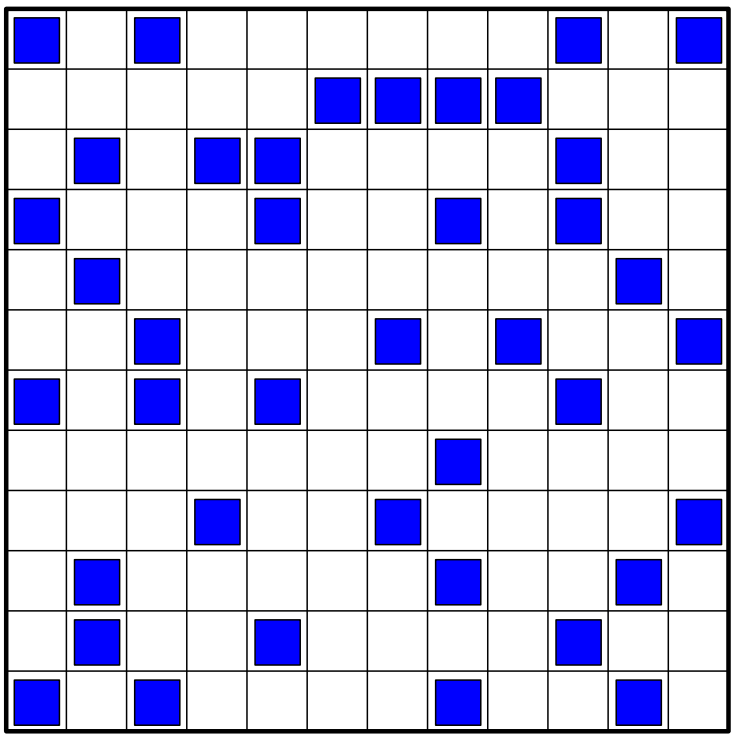

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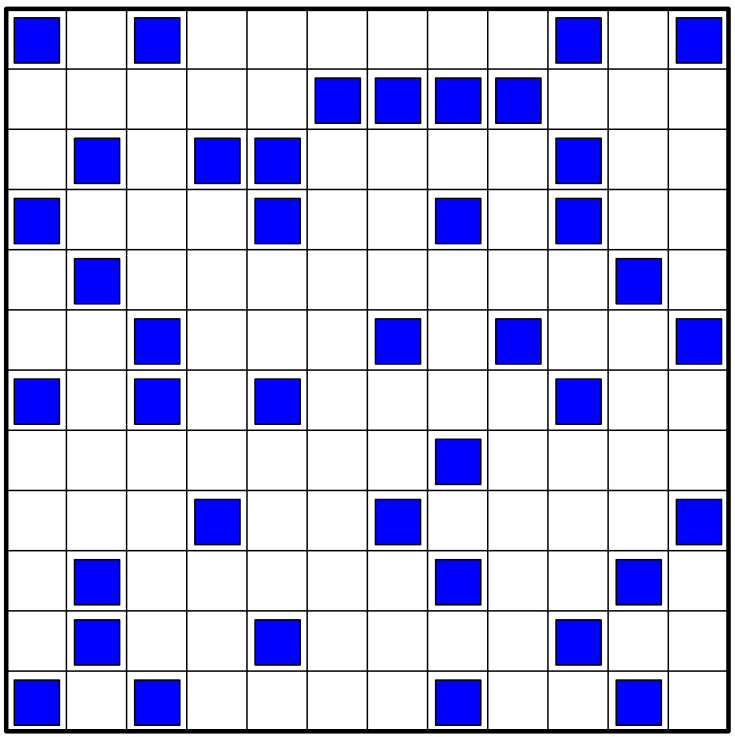

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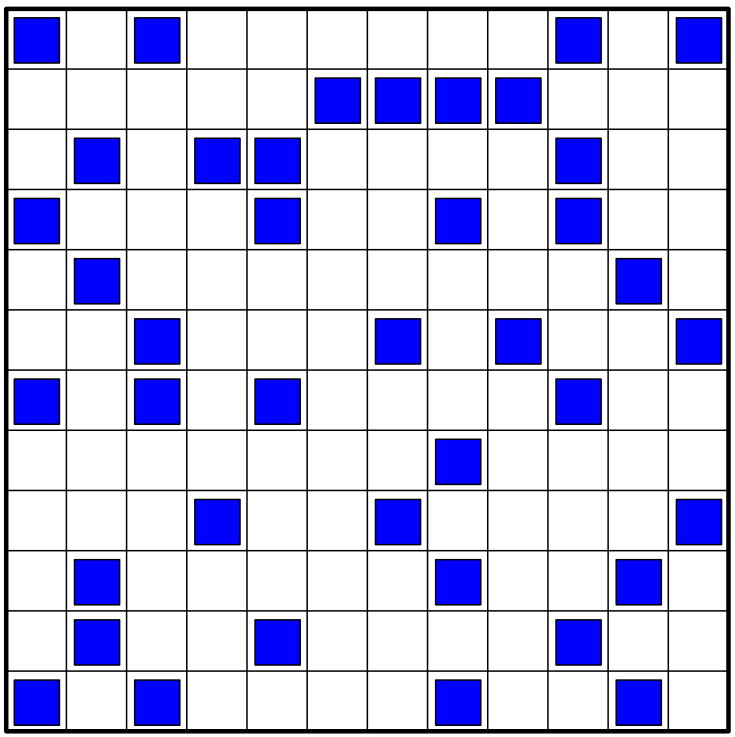

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How to generalise?

Big corner-free set $\Rightarrow$ good protocol

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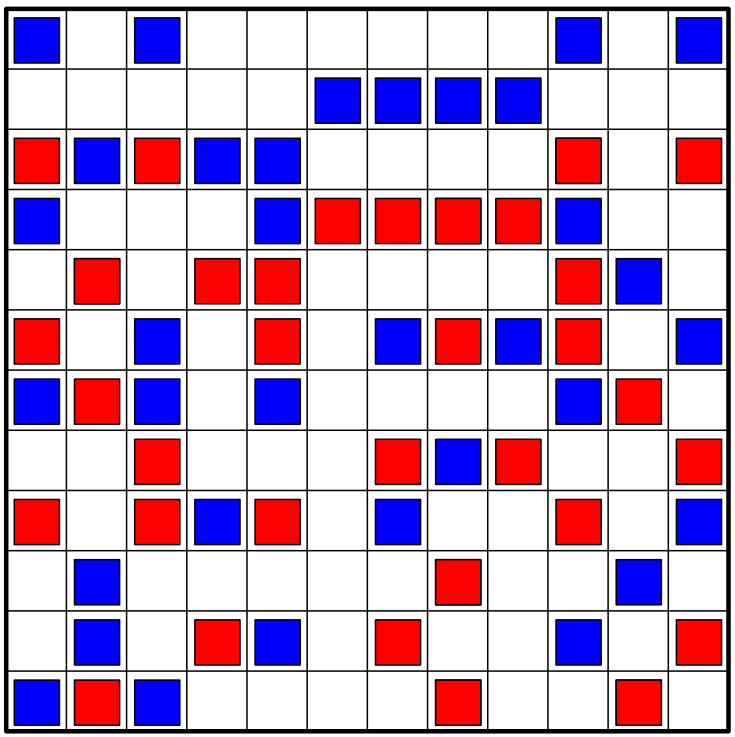

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Shift S! here $S' = S - (0, 2)$

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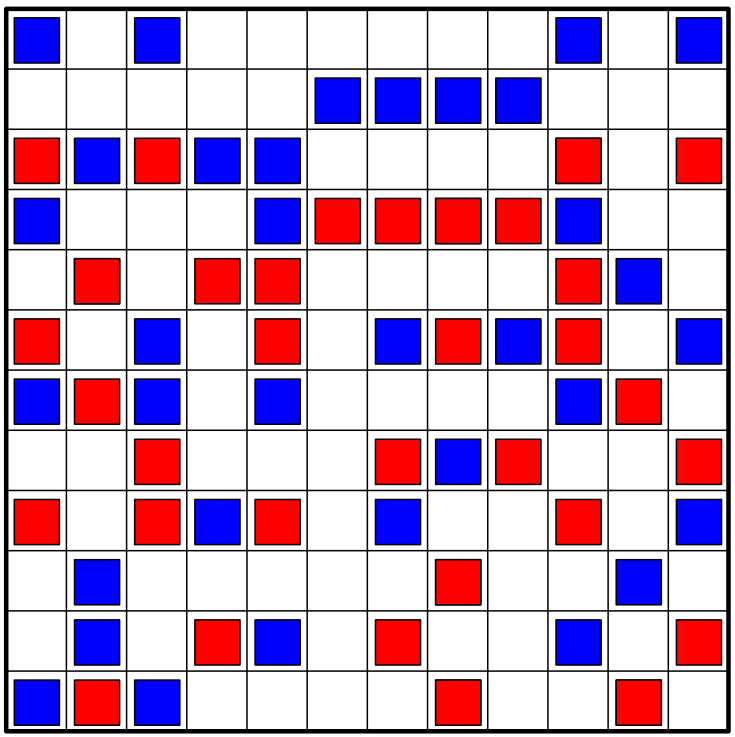

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Such that $[n]^2 \in \cup_{v \in A} S + v$

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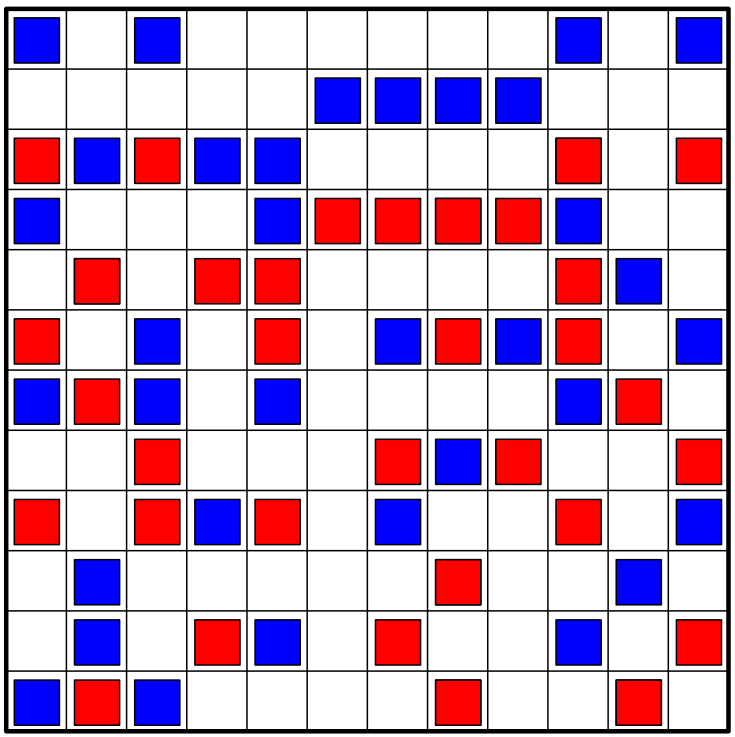

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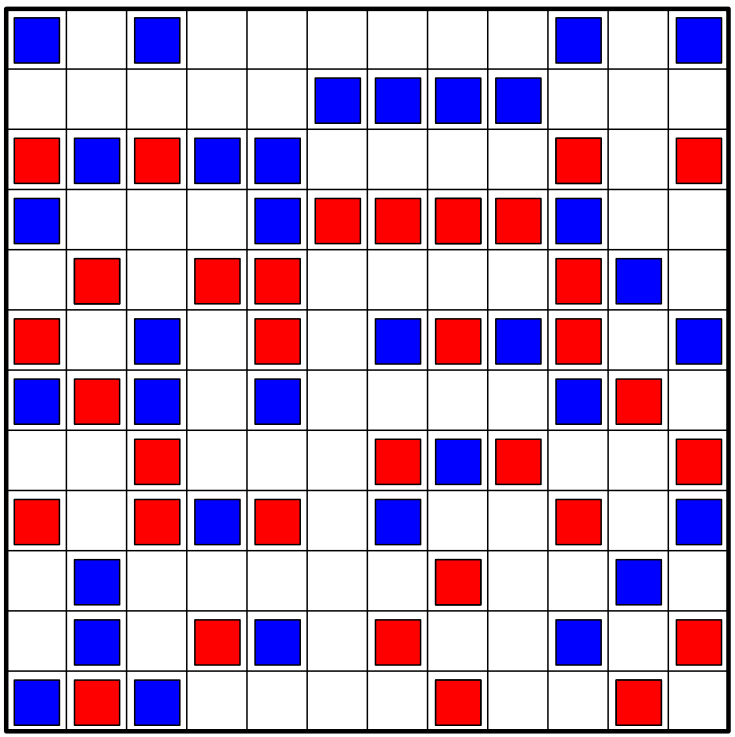

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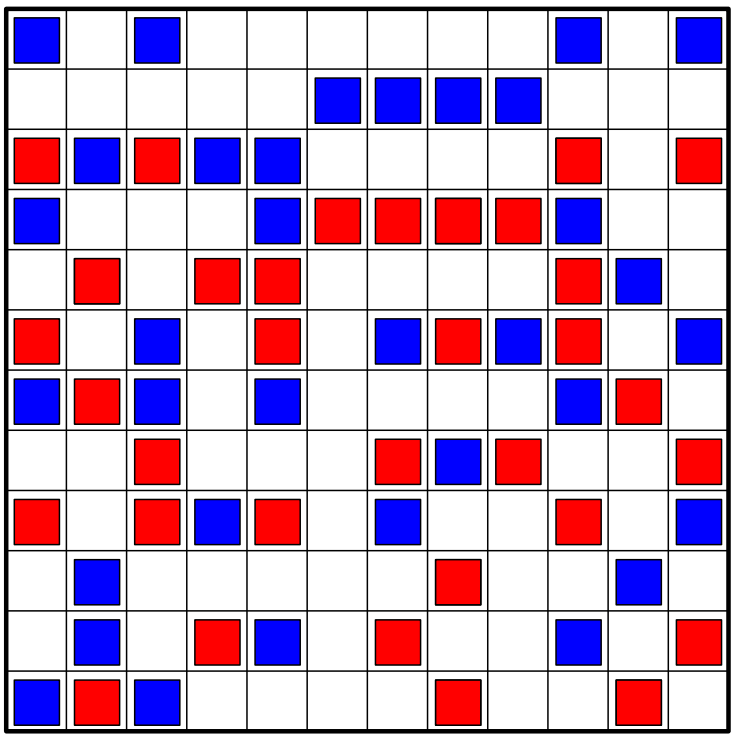

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Complexity:
$\log(\#$ shifts to cover $[n]^2)$
$= \log |A|$

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Let S be a corner-free set of size k

Goal: cover $[n]^2$ with minimum number of shifts of S

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$\rightarrow$ set-cover $\leq O(n^2/k \log n)$

$\rightarrow$ protocol complexity $O(\log(n/k))$

Summary

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If a one round NOF equal protocol with $\leqslant b$-bits exists,
then there is a corner-free set of density $1/2^b$.

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If a corner-free set of density $1/2^b$ exists,
then a one round NOF equal protocol with $\leqslant O(\log(1/2^b)) = O(b)$-bits exists.

Theorem[Alon, Shraibman, '20]

There is a NOF equal protocol on $\leqslant \theta(\sqrt{\log n})$-bits.

A simple protocol on $[\mathbb{Z}_{2q}^d]^2$ instead of $[n]^2$

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We work with vectors $\overline{x} := (x_1, \ldots, x_d) \in \mathbb{N}^d \quad x_i \leq 2q$

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Complexity $d + 2\log dq = \theta(\sqrt{\log n})$ for good d, q

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Thank you!