

# The Sensitivity Conjecture

Journées CALAMAR

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Malory Marin

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ENS de Lyon, France

## The conjecture

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## Conjecture (Sensitivity Conjecture, Nisan and Szgedy, '92)

*There exists an absolute constant  $C > 0$  such that for every boolean function  $f$ ,*

$$bs(f) \leq s(f)^C$$

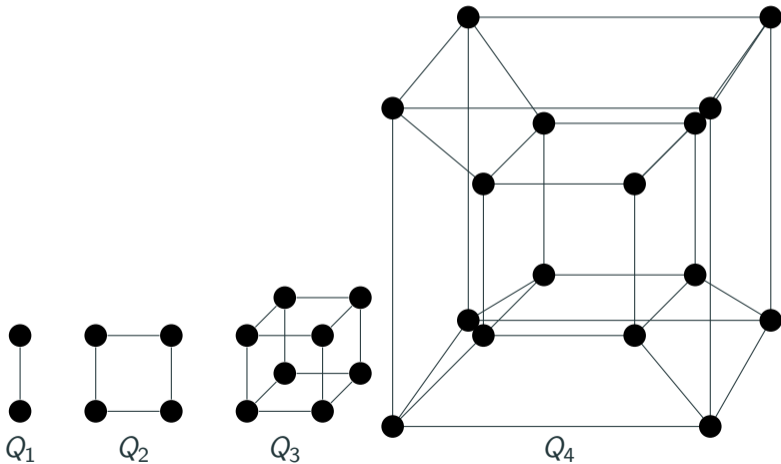
- $s(f)$  : **sensitivity of  $f$**  ;
- $bs(f)$  : **block sensitivity of  $f$**

## An equivalent formulation

Let  $Q_n$  be the hypercube of dimension  $n$ .

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### **Theorem (Gotsman and Linial, '92)**

*The sensitivity conjecture is equivalent the following statement. There exists a universal constant  $C > 0$  such that, for any induced subgraph  $H$  of  $Q_n$  with  $|V(H)| \neq 2^{n-1}$ , we have*

$$\max\{\Delta(H), \Delta(G - H)\} \geq n^C$$

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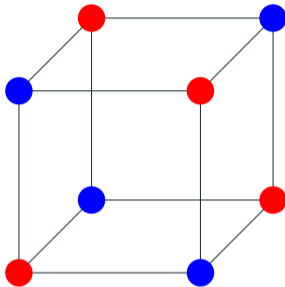
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### **Theorem (Huang, '19)**

*For every integer  $n \geq 1$ , and for every induced subgraph  $H$  of  $Q_n$  with  $2^{n-1} + 1$  vertices, it holds*

$$\Delta(H) \geq \sqrt{n}$$

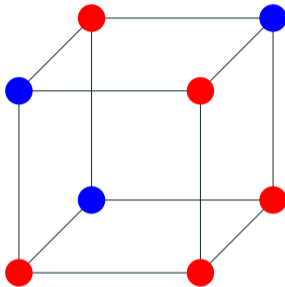
## The 3-dimensional hypercube $Q_3$



$H = \text{even vertices } (2^{3-1} = 4 \text{ vertices})$

$$\Delta(H) = 0.$$

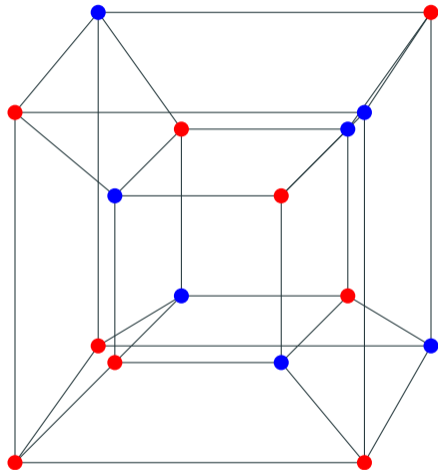
## The 3-dimensional hypercube $Q_3$



$H =$  even vertices  $+1$  vertex ( $2^{3-1} + 1 = 5$  vertices)

$$\Delta(H) = 3 > \sqrt{3}.$$

# The 4-dimensional hypercube $Q_4$



$H =$  red vertices,  $2^{4-1} + 1$  vertices,  $\Delta(H) = 4$

The conjecture

More context : boolean complexity measures

The proof of Huang's theorem

Conclusion

**More context : boolean complexity  
measures**

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# Many Ways to Measure Complexity

Take a boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ .

*How difficult is it to compute a Boolean function?*

There is no unique notion of complexity.

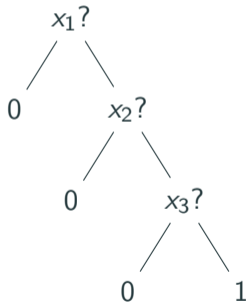
Different computational models lead to different measures:

- ▶ Decision tree complexity,
- ▶ Polynomial degree,
- ▶ Quantum query complexity,
- ▶ Block sensitivity,
- ▶ Sensitivity.

## ▶ Decision Tree Complexity

Consider  $f(x_1, x_2, x_3) = x_1 \wedge x_2 \wedge x_3$ .

To determine the value of  $f$ , we may query variables one by one.



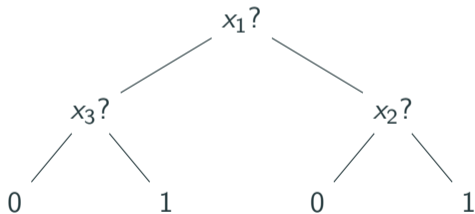
Worst-case number of queries:

$$D(f) = 3$$

## ▶ Decision Tree Complexity

Consider  $f(x_1, x_2, x_3) = x_1 \wedge x_2 \vee \bar{x}_1 \wedge x_3$ .

To determine the value of  $f$ , we may query variables one by one.



Worst-case number of queries:

$$D(f) = 2$$

- ▶ Decision Tree complexity

## ► Polynomial Degree

Every Boolean function admits a unique multilinear polynomial representation.

For example,  $f(x_1, x_2, x_3) = x_1x_2x_3$ .

The largest monomial has degree 3.

$$\deg(f) = 3$$

## ► Polynomial Degree

Every Boolean function admits a unique multilinear polynomial representation.

Another example,

$$x_1 \oplus x_2 \oplus x_3 = x_1 + x_2 + x_3 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 4x_1x_2x_3.$$

Again,

$$\deg(f) = 3$$

## ► Polynomial Degree

Every Boolean function admits a unique multilinear polynomial representation.

### **Theorem**

*For any boolean function  $f$ ,  $\deg(f) \leq D(f)$ .*

### **Proof.**

Transform the decision tree into a polynomial of degree  $D(F)$ . □

### **Theorem (Nisan–Szegedy, 1992)**

$D(f) = O(\deg(f)^3)$ .

## Some polynomial equivalences

- ▶ Decision Tree complexity
- ▶ Polynomial Degree

## ► Sensitivity

Given  $f$ ,

$$s(f) = \max_{x \in \{0,1\}^n} \underbrace{\max \{ |I| \mid I \subseteq [n], \forall i \in I, f(x^{\oplus i}) \neq f(x) \}}_{:= s(f,x)}$$

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For example, let  $f(x) = x_1 \vee x_2 \vee \dots \vee x_n$ .

At the input  $(0, \dots, 0)$ , flipping any coordinate changes the value.

$$s(f, (0, \dots, 0)) = n.$$

Hence

$$s(f) = n$$

Sensitivity counts how many *single-bit changes* can affect the output.

## Some polynomial equivalences

- ▶ Decision Tree complexity
- ▶ Polynomial Degree

- ▶ Sensitivity

## ► Block Sensitivity

Given  $f$ ,

$$bs(f) = \max_{x \in \{0,1\}^n} \underbrace{\max \left\{ k \mid \exists \text{ disjoint blocks } B_1, \dots, B_k \subseteq [n], \forall j, f(x^{\oplus B_j}) \neq f(x) \right\}}_{:= bs(f,x)}$$

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### Observation

For any  $f$ ,  $bs(f) \geq s(f)$ .

### Theorem (Nisan–Szegedy, 1992)

*Block sensitivity is polynomially equivalent to the Decision tree complexity.*

## Some polynomial equivalences

- ▶ Decision Tree complexity
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Sensitivity Conjecture !

In 2019, Hao Huang proved the Sensitivity Conjecture.

**Conjecture (Sensitivity Conjecture, Nisan and Szgedy, '92)**

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## The proof of Huang's theorem

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## The theorem and Previous result

Let  $Q_n$  be the  $n$ -dimensional hypercube.

### **Theorem (Huang, '19)**

*For every integer  $n \geq 1$ , and for every induced subgraph  $H$  of  $Q_n$  with  $2^{n-1} + 1$  vertices, it holds*

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### **Theorem (Chung, Füredi, Graham, Seymour, '88)**

1.  $Q_n$  has a  $(2^{n-1} + 1)$ -vertex induced subgraph of maximum degree  $\lceil \sqrt{n} \rceil$ . **LOWER BOUND**
2. Every  $(2^{n-1} + 1)$ -vertex induced subgraph of  $Q_n$  has maximum degree at least  $(\frac{1}{2} - o(1)) \log_2(n)$ . **UPPER BOUND**

## Idea 1

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### **Lemma**

Let  $H$  be a graph (at least one edge) with adjacency matrix  $A$ . Then,

$$\Delta(H) \geq \lambda_1(A)$$

where  $\lambda_1$  is the largest eigenvalue of  $A$ .

### **Proof.**

Let  $v = (v_1, \dots, v_m)$  be an eigenvector corresponding to  $\lambda_1$ . Wlog, the largest entry of  $v$  is  $v_1$ .

$$|\lambda_1 v_1| = |(Av)_1| = \left| \sum_{j=1}^m A_{1,j} v_j \right| \leq \sum_{j=1}^m |A_{1,j}| |v_j| \leq \Delta(H) |v_1|$$

Thus,  $|\lambda_1| \leq \Delta(H)$ .

**Lemma**

Suppose  $H$  is an  $m$ -vertex undirected graph, and  $M$  is a symmetric matrix whose entries are in  $\{-1, 0, 1\}$  and whose rows and columns are indexed by  $V(H)$ , and whenever  $u$  and  $v$  are non-adjacent in  $H$ ,  $M_{u,v} = 0$ . Then

$$\Delta(H) \geq \lambda_1(M).$$

- Now, we want to choose  $-1$  and  $1$  entries for the adjacency matrix of  $Q_n$
- We want large eigenvalue ( $\geq \sqrt{n}$ ) for any induced subgraph of size  $2^{n-1} + 1$ .

## Idea 2 : Cauchy Entrelacing Theorem

- Suppose that we have a "good" signed matrix  $M$  with large eigenvalues, how to get a bound on the eigenvalues of ANY induced subgraph ?



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### **Lemma**

Let  $M \in \mathbb{R}^{N \times N}$  be a symmetric matrix, and let  $M_S$  be a principal submatrix obtained by restricting to a subset  $S \subseteq [N]$  with  $|S| = m$ . Let:

- $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$  be the eigenvalues of  $M$  ;
- $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$  be those of  $M_S$ .

For every  $1 \leq i \leq m$ ,

$$\lambda_i \geq \mu_i \geq \lambda_{N-m+i}.$$

Take  $|S| = 2^{n-1} + 1$ , we have :

$$\mu_1 \geq \lambda_{2^{n-1}}$$

Objective : find  $M$  signed matrix of  $Q_n$  such that

$$\lambda_1 \geq \dots \geq \lambda_{2^{n-1}} \geq \sqrt{n}$$

## The signed matrix

Define by induction:

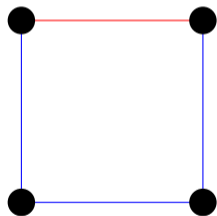
$$M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_n = \begin{pmatrix} M_{n-1} & I \\ I & -M_{n-1} \end{pmatrix}$$



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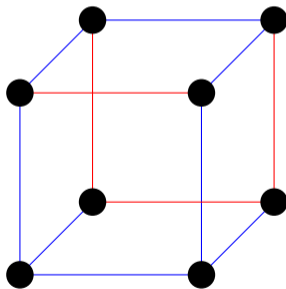
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### Lemma

$M_n$  is a signed adjacency matrix of  $Q_n$  and  $M_n^2 = nI$ .

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### Proof.

By induction.

$$M_n^2 = \begin{pmatrix} M_{n-1}^2 + I & M_{n-1} - M_{n-1} \\ M_{n-1} - M_{n-1} & M_{n-1}^2 + I \end{pmatrix} = nI$$



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## Lemma

$$\text{Spec}(M_n) = \underbrace{\{\sqrt{n}, \dots, \sqrt{n}\}}_{2^{n-1} \text{ times}}, \underbrace{\{-\sqrt{n}, \dots, -\sqrt{n}\}}_{2^{n-1} \text{ times}}.$$

## Proof.

Let  $\lambda$  be an eigenvalue of  $M_n$  with eigenvector  $v$  :  $|M_n^2 v| = |\lambda_1^2 v| = n|v|$ . Thus,  $|\lambda_1| = \sqrt{n}$ . Since  $\text{Tr}(M_n) = 0$ , half of the eigenvalues are  $\sqrt{n}$ , and the other half is  $-\sqrt{n}$ .

- Let  $H$  be an induced subgraph of  $Q_n$  with  $2^{n-1} + 1$  vertices, and  $M_H$  the principal submatrix of  $M_n$  corresponding to the vertices of  $H$ .

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- By Cauchy entrelacing theorem :

$$\mu_1 \geq \lambda_{2^{n-1} - (2^{n-1} + 1) + 1} = \lambda_{2^{n-1}}$$

where  $\lambda_i$  is the  $i$ th largest eigenvalue of  $M_n$ , and  $\mu_1$  the largest eigenvalue of  $H$ .

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- We have  $\lambda_{2^{n-1}} = \sqrt{n}$  and  $\Delta(H) \geq \mu_1$ .
- Altogether :

$$\Delta(H) \geq \sqrt{n}$$

## Conclusion

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# The Aanderaa–Karp–Rosenberg Conjecture

## Graph Properties

Let  $f : \{0, 1\}^{\binom{n}{2}} \rightarrow \{0, 1\}$  be a Boolean function representing a graph property:

- the variables correspond to the possible edges of a graph on  $n$  vertices;
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## Aanderaa-Karp-Rosenberg Conjecture

For any monotone non-trivial graph property  $f$ ,  $D(f) = \binom{n}{2}$ .

## Status:

- Proven when the number of vertices is a prime power (Kahn-Saks-Sturtevant '84).
- Bound of  $\Omega(n^2)$  (Rivest-Vuillemin).