Ramsey's Theorem, Frankl-Wilson Theorem and applications

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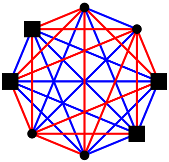
Ramsey Theorem

Theorem (2-colored Ramsey's Theorem)

There exists $R(s,t) \in \mathbb{N}$ such that for every $N \geqslant R(s,t)$, in every $\{\mathtt{red}, \mathtt{blue}\}$-edge-coloring of K_N, there is a red copy of K_s, or a blue copy of K_t.



Proof: induction on $s + t$.
Base case: $s \leqslant 2$ or $t \leqslant 2 \rightarrow$ trivial.
Induction: Take $R(s,t) = R(s-1,t) + R(s,t-1)$. For $u \in V(K_N)$, either $d_{\mathtt{red}}(u) \geqslant R(s-1,t)$ or $d_{\mathtt{blue}}(u) \geqslant R(t,s-1)$. In both cases, we win. □

Bound: $R(s,t) \leqslant \binom{s+t-1}{s-2}$.

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Theorem (k-colored Ramsey's Theorem)

There exists $R_k(t) \in \mathbb{N}$ such that for every $N \geqslant R_k(t)$, in every k-edge-coloring of K_N, there is a monochromatic copy of K_t.

Proof: $R_k(t) \leqslant R(t, R_{k-1}(t))$. □

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Proof: $R_k(t) \leqslant R(t, R_{k-1}(t))$. □

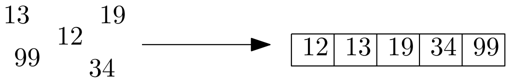
Theorem (k-colored m-dimensional Ramsey's Theorem)

There exists $R_k(t;m) \in \mathbb{N}$ such that for every $N \geqslant R_k(t;m)$, in every k-coloring of $\binom{[N]}{m}$, there is a monochromatic subset of size t.

Proof: induction on m. □

A first application: a theorem of Yao

Problem: storing $S \subseteq [M]$ of size n into an n-array such that the queries "$i \mapsto i \in S$?" can be answered quickly.
Measure of the complexity: number of readings of the array in the worst case.
First approach: storing S in increasing order. Binary search gives $\lceil \log(n+1) \rceil$.



Second approach: Hash tables (but $\Theta(n)$ in the worst case).

Theorem (Yao; 1981)

If M is large enough compared to n, then the Sorted Table has optimal worst-case complexity.

Proof of Yao's theorem

Theorem (Yao; 1981)

If M is large enough compared to n, then the Sorted Table has optimal worst-case complexity.

Notation: the algorithm is a function $A\colon \binom{[M]}{n} \to [M]^n$ such that $A(S)$ is a permutation of S.

First step: color every $S \in \binom{[M]}{n}$ by the ordering of $A(S)$
$\to$ at most $n!$ colors.

Ramsey $\Rightarrow$ there exists $U \subseteq [M]$ of size $2n-1$ such that A sorts n-subsets of U always in the same order.

W.l.o.g $U = [2n-1]$ and A sorts the subsets of $[2n-1]$ in increasing order.

Lemma

In this case, $\lceil \log(n+1) \rceil$ readings are needed to answer "$n \in S$?".

Proof: standard adversary construction. □

Upper bounds for diagonal Ramsey numbers

Diagonal Ramsey number: $R(t,t)$.

Erdős-Moser: $R(t,t) \leqslant \binom{2t-2}{t-1} \leqslant 4^t$.

For a long time only subexponential improvement, and then:

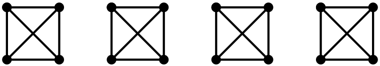
Theorem (Campos, Griffiths, Morris, Sahasrabudhe; 2023)

$$R(t,t) \leqslant (4-\epsilon)^t.$$

Current best: $R(t,t) \leqslant 3.7992^{t+o(t)}$.
(Gupta, Ndiaye, Norin, and Wei; 2024)

Lower bounds for diagonal Ramsey numbers

A first construction: $R(t,t) \geqslant (t-1)^2 + 1$.



Theorem (Erdős, Szekeres; 1935)

$$R(t,t) \geqslant 2^{t/2 - o(t)}.$$

Proof: color K_N at random.

$$\begin{aligned}
&\Pr[\text{there is a monochromatic } t\text{-subgraph}] \\
&\leqslant \sum_{S \subseteq V(K_N), |S|=t} \Pr[S \text{ monochromatic}] \\
&\leqslant \binom{N}{t} \cdot 2 \cdot 2^{-\binom{t}{2}} \\
&\leqslant 2^{1 + t\log N - t(t-1)/2} < 1. \quad \square
\end{aligned}$$

A (big) detour: extremal set theory

General question:

How big can be $\mathcal{F} \subseteq 2^{[n]}$ satisfying some properties ?

Examples:

- members of $\mathcal{F} \subseteq 2^{[n]}$ are pairwise incomparable
  $\rightarrow \quad |\mathcal{F}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$ (Sperner's Theorem)
- members of $\mathcal{F} \subseteq \binom{[n]}{k}$ pairwise intersect
  $\rightarrow \quad |\mathcal{F}| \leqslant \binom{n-1}{k-1}$ (Erdős-Ko-Rado Theorem)
- at most s intersections sizes between members of $\mathcal{F} \subseteq \binom{[n]}{k}$
  $\rightarrow \quad |\mathcal{F}| \leqslant \binom{n}{s}$ (Frankl-Wilson Theorem)
- no shattered $(d+1)$-subset
  $\rightarrow \quad |\mathcal{F}| \leqslant \sum_{i=0}^{d} \binom{n}{i}$ (Sauer-Shelah Lemma)

Erdős-Ko-Rado Theorem

Theorem (Erdős-Ko-Rado Theorem; 1961)

Let $n \geqslant 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$. If $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$, then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1}.$$

Tight: $\mathcal{F} = \{F \in \binom{[n]}{k} \mid 1 \in F\}$.

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In terms of graphs:

For the **Kneser Graph** $K(n,k) = \left(\binom{[n]}{k}, \{FF' \mid F \cap F' = \emptyset\}\right)$,

$$\alpha(K(n,k)) \leqslant \binom{n-1}{k-1} = \frac{k}{n}|V(K(n,k))|.$$

An averaging technique

Vertex-transitive: $\forall x, y \in V(G), \exists \tau \in \mathrm{Aut}(G), \tau(x) = y$.
$\rightarrow K(n,k)$ is vertex-transitive.

Lemma

Let G be a vertex-transitive graph. For every $H \subseteq G$,

$$\frac{\alpha(G)}{|V(G)|} \leqslant \frac{\alpha(H)}{|V(H)|}.$$

For H a maximum clique: $\alpha(G)\omega(G) \leqslant |V(G)|$.

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Proof: Let σ be drawn at random in $\mathrm{Aut}(G)$.
Claim: for all u, $\sigma(u)$ follows a uniform distribution over $V(G)$.
Proof: for all $x, y \in V(G), \exists \tau \in Aut(G)\ s.t.\ \tau(x) = y$, so

$$\Pr[\sigma(u) = x] = \Pr[(\tau \circ \sigma)(u) = y] = \Pr[\sigma(u) = y].$$

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For every independent set S in G

$$\alpha(H) \geqslant \mathbb{E}[|V(H) \cap \sigma(S)|] = \sum_{u \in S} \Pr[\sigma(u) \in V(H)] = |S| \frac{|V(H)|}{|V(G)|}. \quad \square$$

Proof of the Erdős-Ko-Rado Theorem

Goal:

$$\frac{\alpha(K(n,k))}{|V(K(n,k))|} \leqslant \frac{k}{n}.$$

A first simple case: $k|n$. Take H be an (n/k)-clique:

$$\frac{\alpha(K(n,k))}{|V(K(n,k))|} \leqslant \frac{\alpha(H)}{|V(H)|} = \frac{k}{n}.$$

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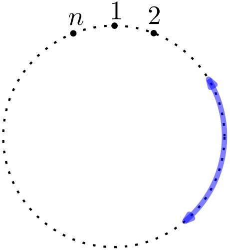
General case (due to Katona):
Take H induced by $\{[i, i+k-1] \mid i \in [n]\}$ (modulo n).



$$|V(H)| = n$$
$$\alpha(H) = k.$$

□

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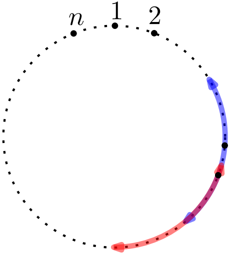

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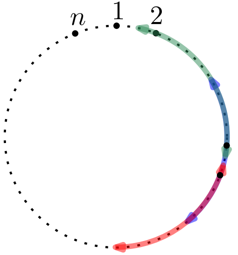

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□

Only one intersection size

Theorem

Let $\mathcal{F} \subseteq \binom{n}{k}$. If for all $F, F' \in \mathcal{F}$ distinct $|F \cap F'| = t$, then

$$|\mathcal{F}| \leqslant n.$$

Proof: Consider the $n \times |\mathcal{F}|$ matrix M defined by

$$M_{x,F} = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$M^\top \cdot M = \begin{pmatrix} k & t & \cdots & t \\ t & k & \cdots & t \\ \vdots & \vdots & \ddots & \vdots \\ t & & \cdots & k \end{pmatrix} \qquad \text{so} \qquad \begin{aligned} n &\geqslant \operatorname{rk} M \\ &\geqslant \operatorname{rk}(M^\top \cdot M) \\ &\geqslant |\mathcal{F}|. \end{aligned}$$

□

Only one intersection size (modulo 2)

Theorem

Let $\mathcal{F} \subseteq 2^{[n]}$. If

1. $|F|$ odd for all $F \in \mathcal{F}$, and
2. $|F \cap F'|$ even for all $F, F' \in \mathcal{F}$ distinct,

then $|\mathcal{F}| \leqslant n$.

Proof: Consider the $n \times |\mathcal{F}|$ matrix M defined by

$$M_{x,F} = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{otherwise.} \end{cases}$$

Then, in $\mathbb{F}_2$,

$$M^\top \cdot M = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & \cdots & 1 \end{pmatrix} \qquad \text{so} \qquad \begin{aligned} n &\geqslant \operatorname{rk} M \\ &\geqslant \operatorname{rk}(M^\top \cdot M) \\ &\geqslant |\mathcal{F}|. \end{aligned}$$

□

A cubic constructive lower bound for Ramsey numbers

Construction due to (Nagy; 1975):

$$V(G) = \binom{[n]}{3}$$
$$E(G) = \{XY \mid |X \cap Y| \equiv 1 \bmod 2\}.$$

Claim: $\alpha(G), \omega(G) \leqslant n$

Proof:

- an idendependent set is a family $\mathcal{F} \subseteq \binom{[n]}{3}$ such that $|F \cap F'| \equiv 0 \mod 2$ for every $F, F' \in \mathcal{F}$ distinct.
- a clique is a family $\mathcal{F} \subseteq \binom{[n]}{3}$ such that $|F \cap F'| = 1$ for every $F, F' \in \mathcal{F}$ distinct.

This gives

$$R(n+1, n+1) \geqslant \binom{n}{3} + 1.$$

Frankl-Wilson Theorem

Theorem (Frankl-Wilson Theorem; 1981)

Let p be a prime, and let $k, \mu_1, \ldots, \mu_s$ distinct modulo p. For every $\mathcal{F} \subseteq \binom{n}{k}$ such that $|F \cap F'| \bmod p \in \{\mu_1, \ldots, \mu_s\}$ for all $F, F' \in \mathcal{F}$ distinct, then

$$|\mathcal{F}| \leqslant \binom{n}{s}.$$

Proof:

Let $Q(T) = \prod_{i=1}^{s}(T - \mu_i)$, and M the $|\mathcal{F}| \times |\mathcal{F}|$ matrix

$$M_{F,F'} = Q(|F \cap F'|).$$

Over $\mathbb{F}_p$: $M = \begin{pmatrix} Q(k) & 0 & \cdots & 0 \\ 0 & Q(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & \cdots & Q(k) \end{pmatrix}$

Then, M has full rank over $\mathbb{F}_p$, and so over $\mathbb{Q}$: $\operatorname{rk} M = |\mathcal{F}|$.

Proof of the Frankl-Wilson Theorem

$Q(T) = \prod_{i=1}^{s}(T - \mu_i)$ is a polynomial of degree s, so there exists $a_1, \ldots, a_s \in \mathbb{Q}$ such that $Q(T) = \sum_{j=0}^{s} a_j \binom{T}{j}$.

$$
\begin{aligned}
M_{F,F'} &= \sum_{j=0}^{s} a_j \binom{|F \cap F'|}{j} = \sum_{j=0}^{s} a_j \sum_{Z \in \binom{F}{j}} [Z \subseteq F'] \\
&= \sum_{j=0}^{s} a_j \sum_{Z \in \binom{F}{j}} \sum_{Z' \in \binom{n}{s}} \frac{[Z \subseteq Z'][Z' \subseteq F']}{\binom{k-j}{s-j}} \\
&= \sum_{Z' \in \binom{[n]}{s}} \left(\sum_{j=0}^{s} a_j \frac{|\{Z \in \binom{F}{j} \mid Z \subseteq Z'\}|}{\binom{k-j}{s-j}} \right) [Z' \subseteq F'] \\
&= \sum_{Z' \in \binom{[n]}{s}} c_{F,Z'} [Z' \subseteq F'].
\end{aligned}
$$

So every row M_F is in the span of the vectors $([Z' \subseteq F'])_{F' \in \mathcal{F}}$ for $Z' \in \binom{[n]}{s}$. So $|\mathcal{F}| = \mathrm{rk}\, M \leqslant \binom{n}{s}$. □

Applications of the Frankl-Wilson Theorem

Theorem (Frankl-Wilson Theorem)

Let p be a prime, and let $k, \mu_1, \ldots, \mu_s$ distinct modulo p. For every $\mathcal{F} \subseteq \binom{n}{k}$ such that $|F \cap F'| \bmod p \in \{\mu_1, \ldots, \mu_s\}$ for all $F, F' \in \mathcal{F}$ distinct, then $|\mathcal{F}| \leqslant \binom{n}{s}$.

Corollary

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and $L \subseteq \{0, \ldots, k-1\}$ of size s. If $|F \cap F'| \in L$ for all distinct $F, F' \in \mathcal{F}$, then $|\mathcal{F}| \leqslant \binom{n}{s}$.

Proof: take p large enough.

Example: taking $L = \{1, \ldots, k-1\}$ gives almost Erdős-Ko-Rado.

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Construction: Consider G_p with vertex set $\binom{[p^3]}{p^2-1}$ and edges XY for $|X \cap Y| \in \{p-1, 2p-1, \dots, (p-1)p-1\}$ (i.e. $\equiv -1 \mod p$).

- Corollary $\Rightarrow \omega(G_p) \leqslant \binom{p^3}{p-1}$
- F-W $\Rightarrow \alpha(G_p) \leqslant \binom{p^3}{p-1}$

$$\Rightarrow R\left(\binom{p^3}{p-1}+1, \binom{p^3}{p-1}+1\right) > \binom{p^3}{p^2-1}$$

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- Corollary $\Rightarrow \omega(G_p) \leqslant \binom{p^3}{p-1}$
- F-W $\Rightarrow \alpha(G_p) \leqslant \binom{p^3}{p-1}$

$\Rightarrow R(t,t) > t^{\Omega(\log t / \log\log t)}$

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Corollary

Let $\mathcal{F} \subseteq \binom{[4p]}{2p}$. If $|F \Delta F'| \neq 2p$ for every $F, F' \in \mathcal{F}$, then

$$|\mathcal{F}| \leqslant 2\binom{4p}{p-1} \leqslant (2-\epsilon)^{4p}.$$

Proof: First, $|F \Delta F'| \neq 2p$ iff $|F \cap F'| \neq p$. For each pair of the form $\{F, [4p] \setminus F\} \subseteq \mathcal{F}$, keep only one member. This gives $\mathcal{F}'$ such that $|\mathcal{F}| \leqslant 2|\mathcal{F}'|$ and $|F \cap F'| \notin \{0, p\}$ for all $F, F' \in \mathcal{F}'$ distinct.

Frankl-Wilson $\Rightarrow |\mathcal{F}| \leqslant 2|\mathcal{F}'| \leqslant 2\binom{4p}{p-1}$. □

Some geometric applications

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Let $\mathcal{F} \subseteq \{-1,+1\}^{4p}$. If $\vec{x} \cdot \vec{y} \neq 0$ for every $\vec{x}, \vec{y} \in \mathcal{F}$, then

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Corollary

Let X be a measurable subset of $\mathcal{S}^{4p}$. If X does not contains two orthogonal vectors, then X has measure at most $(1-\epsilon)^{4p}$.

Proof: averaging argument for H induced by $\{\pm 1\}$-vectors. □

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Corollary

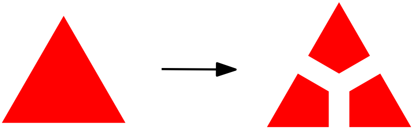
The graph with vertex set $\mathbb{R}^n$ and edges set $\{xy \mid \mathrm{dist}(x,y) = 1\}$ has chromatic number at least $(1+\epsilon)^n$.

Proof: approximate n with $4p$, then the $\{\pm\frac{1}{\sqrt{2n}}\}$-vectors induces a subgraphs with too small α. □

A last geometric application: Borsuk's problem

Let $S \subseteq \mathbb{R}^n$ bounded.

What is the smallest $b(S)$ such that S can be covered by $b(S)$ sets of smaller diameter ?



Example: for $S = \{e_1, \ldots, e_n, -\frac{1}{n}\sum e_i\}$, we get $b(S) = n + 1$.

Borsuk's problem: is it always at most $n + 1$?

Theorem (Kahn, Kalai; 1993)

There is a set $S \subseteq \mathbb{R}^n$ such that $b(S) \geqslant c^{\sqrt{n}}$ for some $c > 1$.

The Kahn-Kalai construction

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There is a set $S \subseteq \mathbb{R}^n$ such that $b(S) \geqslant c^{\sqrt{n}}$ for some $c > 1$.

Proof (combinatorial formulation): take $n = \binom{4p}{2}$, and identify subsets of $\binom{[4p]}{2}$ with their $\{-1, +1\}$-vectors.

$$S = \left\{ E(K_{x,\bar{x}}) \mid x \in \binom{[4p]}{2p} \right\}$$

where $E(K_{x,\bar{x}}) = \{\{i, j\} \mid i \in x, j \notin x\}$.

Claim: $\mathrm{dist}(E(K_{x,\bar{x}}), E(K_{y,\bar{y}}))$ is max iff $|x \Delta y| = 2p$.

Frankl-Wilson $\Rightarrow$

if $S' \subseteq S$ has smaller diameter, then $|S'| \leqslant 2\binom{4p}{p-1}$.

$\Rightarrow$ we need at least $\frac{\binom{4p}{2p}}{2\binom{4p}{p-1}} \geqslant (1+\epsilon)^p$ of them to cover S. □

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Theorem (Kahn, Kalai; 1993)

There is a set $S \subseteq \mathbb{R}^n$ such that $b(S) \geqslant c^{\sqrt{n}}$ for some $c > 1$.

Proof (geometric formulation): take $n = \binom{4p}{2}$ and

$$S = \left\{ x \cdot x^\top \mid x \in \left\{ \pm \frac{1}{\sqrt{4p}} \right\}^{4p} \right\} \subseteq \{\text{symmetric } 4p \times 4p \text{ matrices}\}.$$

Then

$$\|x \cdot x^\top - y \cdot y^\top\|^2 = \mathrm{Tr}\left(\left(x \cdot x^\top - y \cdot y^\top \right) \cdot \left(x \cdot x^\top - y \cdot y^\top \right)^\top \right)$$
$$= \|x\|^4 + \|y\|^4 - 2\langle x, y \rangle^2 = 2(1 - \langle x, y \rangle^2)$$

Frankl-Wilson

$\Rightarrow \forall S' \subseteq S, \mathrm{diam}(S') < \mathrm{diam}(S) \Rightarrow |S'| \leqslant (2 - \eta)^{4p}$.

$\Rightarrow$ we need at least $\geqslant (1 + \epsilon)^p$ of them to cover S. $\square$

Frankl-Rödl theorem

Corollary (Frankl-Wilson)

Let $\mathcal{F} \subseteq \binom{[4p]}{2p}$. If $|F \Delta F'| \neq 2p$ for every $F, F' \in \mathcal{F}$, then

$$|\mathcal{F}| \leqslant 2\binom{4p}{p-1} \leqslant (2-\epsilon)^{4p}.$$

Theorem (Frankl-Rödl Theorem)

Let $\mathcal{F} \subseteq 2^{[2n]}$. If $|F \Delta F'| \neq n$ for every $F, F' \in \mathcal{F}$, then

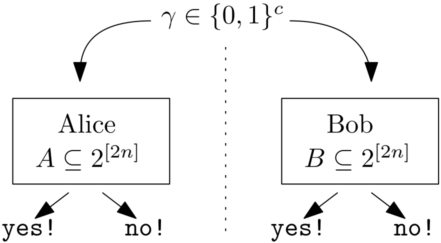
$$|\mathcal{F}| \leqslant (2-\epsilon)^{2n}.$$

Theorem (Cross-intersections Frankl-Rödl Theorem)

Let $\mathcal{F}, \mathcal{G} \subseteq 2^{[2n]}$. If $|F \Delta G| \neq n$ for every $F \in \mathcal{F}, G \in \mathcal{G}$, then

$$|\mathcal{F}| \cdot |\mathcal{G}| \leqslant (4-\epsilon)^{2n}.$$

An application to communication complexity



- Alice receives $A \in 2^{[2n]}$.
- Bob receives $B \in 2^{[2n]}$.

Promise: $|A \Delta B| \in \{0, n\}$. **Goal:** deciding whether $A = B$.

$A = B$ iff $\exists \gamma$ such that both Alice and Bob answer `yes`.

Non-deterministic communication complexity
= minimum c such that this is possible.

An application to communication complexity

Theorem (Buhrman, Cleve, Wigderson; 1998)

1. The non-deterministic communication complexity is in $\Omega(n)$.
2. But it is in $\mathcal{O}(\log n)$ if Alice and Bob can share an *entangled quantum state*.

Proof of the lower bound:

Theorem (Cross-intersections Frankl-Rödl Theorem)

Let $\mathcal{F}, \mathcal{G} \subseteq 2^{[2n]}$. If $|F \Delta G| \neq n$ for every $F \in \mathcal{F}, G \in \mathcal{G}$, then

$$|\mathcal{F}| \cdot |\mathcal{G}| \leqslant (4 - \epsilon)^{2n}.$$

Fix a certificate γ. Let $\mathcal{F} = \{A \mid \text{Alice accepts } A \text{ with } \gamma\}$ and $\mathcal{G} = \{B \mid \text{Bob accepts } B \text{ with } \gamma\}$. We have $|\mathcal{F}| \cdot |\mathcal{G}| \leqslant (4 - \epsilon)^{2n}$.

So γ can cover at most $\sqrt{(4-\epsilon)^{2n}}$ of the 2^{2n} "yes" entries. □

Proof of the Frankl-Rödl theorem

Theorem (Frankl and Rödl; 1987)

For all $\eta \in]0, 1/4[$, there exists $\epsilon > 0$ such that:
For all $d \in [\eta n, (1/2 - \eta)n]$, $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$, if $|F \cap G| \neq d$ for all $F \in \mathcal{F}, G \in \mathcal{G}$, then

$$|\mathcal{F}| \cdot |\mathcal{G}| \leqslant (4 - \epsilon)^n.$$

Sketch of the proof: "density increment technique"

- $\mathcal{P}(n, [a, b]) = \{(\mathcal{F}, \mathcal{G}) \mid \forall F \in \mathcal{F}, G \in \mathcal{G}, |F \cap G| \notin [a, b]\}$
- *density* of $\mathcal{F} \subseteq 2^{[n]}$: $p(\mathcal{F}) = \frac{|\mathcal{F}|}{2^n}$

For $(\mathcal{F}, \mathcal{G}) \in \mathcal{P}(n, [a, b])$, let

- $\mathcal{F}_0 = \{F \in \mathcal{F} \mid n \notin F\}$ and $\mathcal{F}_1 = \{F \setminus \{n\} \mid F \in \mathcal{F}, n \in F\}$
- $\mathcal{G}_0 = \{G \in \mathcal{F} \mid n \notin G\}$ and $\mathcal{G}_1 = \{G \setminus \{n\} \mid G \in \mathcal{G}, n \in G\}$

<u>Claim:</u>

$(\mathcal{F}_1, \mathcal{G}_1) \in \mathcal{P}(n-1, [a-1, b-1])$ $\qquad$ $(\mathcal{F}_0, \mathcal{G}_0 \cup \mathcal{G}_1) \in \mathcal{P}(n-1, [a, b])$

$$(\mathcal{F}_1, \mathcal{G}_0 \cap \mathcal{G}_1) \in \mathcal{P}(n-1, [a-1, b])$$

Sketch of the proof of Frankl-Rödl Theorem

Let $\delta > 0$ small enough.
While $0 < a$ **and** $b < n$**:**

- if $p(\mathcal{F}_1)p(\mathcal{G}_1) \geqslant (1+\delta)p(\mathcal{F})p(\mathcal{G})$, recurse on $(\mathcal{F}_1, \mathcal{G}_1) \in \mathcal{P}(n-1, [a-1, b-1])$
- if $p(\mathcal{F}_0)p(\mathcal{G}_0 \cup \mathcal{G}_1) \geqslant (1+\delta)p(\mathcal{F})p(\mathcal{G})$, recurse on $(\mathcal{F}_0, \mathcal{G}_0 \cup \mathcal{G}_1) \in \mathcal{P}(n-1, [a, b])$
- if $p(\mathcal{F}_1)p(\mathcal{G}_0 \cap \mathcal{G}_1) \geqslant (1-\delta-2\delta^2)p(\mathcal{F})p(\mathcal{G})$, recurse on $(\mathcal{F}_1, \mathcal{G}_0 \cap \mathcal{G}_1) \in \mathcal{P}(n-1, [a-1, b])$

At the end, we get $(\mathcal{F}^*, \mathcal{G}^*) \in \mathcal{P}(n^*, [0, b^*]) \cup \mathcal{P}(n^*, [a^*, n^*])$.

Theorem (Ahlswede and Katona; 1977)

For $\beta > 0$ and $\alpha < 1/2$,

- $(\mathcal{F}^*, \mathcal{G}^*) \in \mathcal{P}(n^*, [0, \beta n^*]) \Rightarrow p(\mathcal{F}^*)p(\mathcal{G}^*) \leqslant (1-\epsilon(\beta))^{n^*}$.
- $(\mathcal{F}^*, \mathcal{G}^*) \in \mathcal{P}(n^*, [\alpha n^*, n^*]) \Rightarrow p(\mathcal{F}^*)p(\mathcal{G}^*) \leqslant (1-\epsilon(\alpha))^{n^*}$.

Thank you !