ε-nets & Discrepancy

I Beck–Fiala

Th For every k-uniform hypergraph $H=(V,E)$ and $w : E \to [0,1]$, $\exists H'=(V,F)$ where $F \subseteq E$ s.t. : $|d_{H'}(v) - d_w(v)| \leq k - \frac{1}{2}$ $\forall v \in V$

Rem: • work for $(\leq k)$-hypergraphs

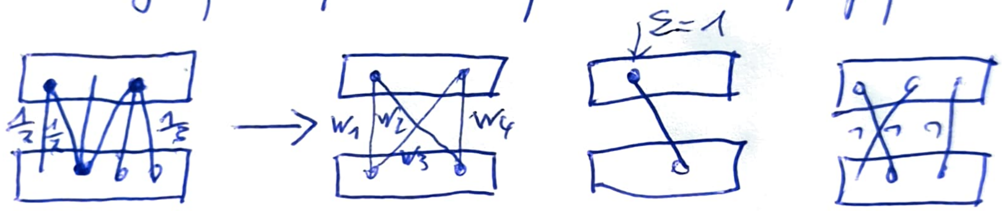
• $\exists w' : E \to \{0,1\}$ / $\|d_{w'} - d_w\|_\infty \leq k - \frac{1}{2}$

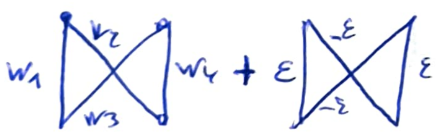
• equivalently $\forall w : E \to \mathbb{R}$ $\exists w' : E \to \mathbb{Z}$ /

$$\|d_w - d_{w'}\|_\infty \leq k - \frac{1}{2}$$

$$\& \; w'(e) = \lfloor w(e) \rfloor \text{ ou } \lceil w(e) \rceil$$

Example: (i) Tout graphe biparti cubique a un couplage parfait





"push" → ∃ arête entière.

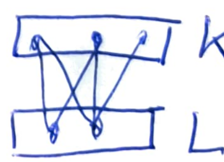
dirc = 0

$0 < \varepsilon < 1$

(ii) Matrix $A \in \mathbb{R}^{k \times l}$ $\exists A' \in \mathbb{Z}^{k \times l}$ / ~~Somme~~

$$\left| \sum_{i=1}^{k} a_{ij} - \sum_{i=1}^{k} a'_{ij} \right| \leq 1 \quad \forall j$$

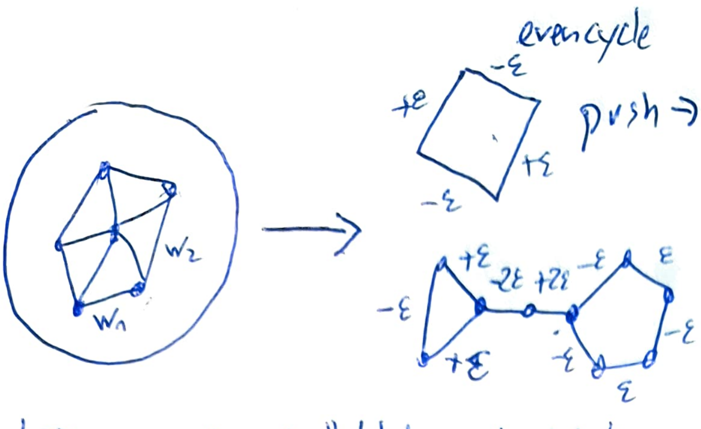
idem $\forall i$



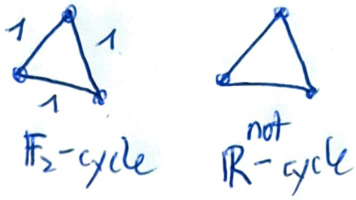
Totally Unimodulaire

(iii) If H is a graph $G=(V,E)$ & $w : E \to [0,1]$

then $\exists H=(V,F)$ s.t. $|d_H(v) - d_w(v)| \leq 1{,}5$



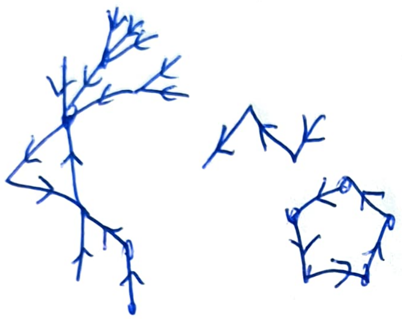
push → integer edge



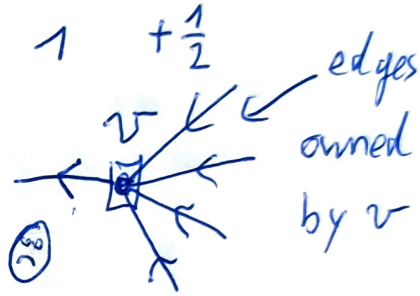
$$\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}$$

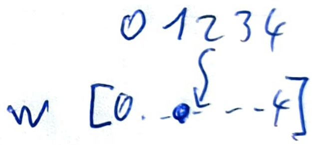
<u>Algo</u>:
- If $w_i = 0$ or 1 "delete and update".
- If $\exists\ \mathbb{R}$-cycle → push.
- If $\exists v$ of degree ≤ 1 "store it".

⇝ G' s.t. all components have at most 1 odd cycle.

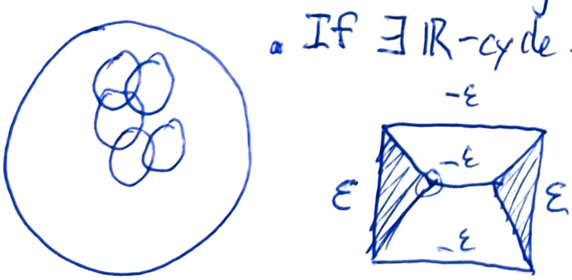


orient with $d^+ \leq 1$





<u>Proof of Beck-Fiala</u>
- If $w_i = 0$ or 1 → delete
- If $\exists v$ of degree $\leq k-1$ → store
- If $\exists\ \mathbb{R}$-cycle → push.

H 

→ H' without $\mathbb{R}$-cycle.

H'
- $d^\circ \geq k$
- $(\leq k)$-uniform.

$\geq k$ "1" and $\exists > k$ "1"

Suppose $\exists v / d^\circ(v) > k$ $\downarrow$ $\boxed{I_H}$

$|V| < |E|$

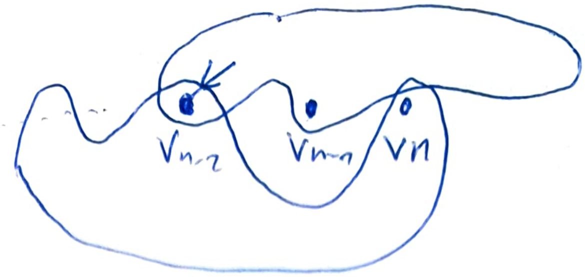
$I_H \cdot x = 0$

$\boxed{H' \text{ k-uniform, k-regular}}$

$E \leftarrow \leq k$ "1" → push.

H'

k-uniform
k-regular
☹ k-1



☹ $\leq k-1$

disc $\leq k-1+\frac{1}{2} = \boxed{k-\frac{1}{2}}$

! Some graph are better than other.

$\boxed{\text{II}}$ VC-dim.

$\boxed{\text{Th}}$ For a class $\mathcal{C}$ of hypergraphs TFAE

- $\mathcal{C}$ strict
- hittingset τ : $\tau \leq f(\tau^*)$ (relax frac) (HW)
- $\exists c \,/\, \forall H=(V,E) \in \mathcal{C}$ $|E| \leq |V|^c$

$\longrightarrow$ VC-dim $(\mathcal{C})$ = min size $H \notin \mathcal{C}$ -1

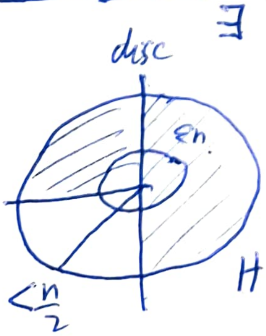
$\boxed{\text{III}}$ Disc

$\boxed{\text{Th}}$ $H=(V,E)$ disc $\leq \sqrt{n \log m}$

$\boxed{\text{Th (Spencer)}}$ disc $\leq \sqrt{n \log \frac{2m}{n}}$

$\boxed{\text{Th (?)}}$ If H has VC-dim d
disc $= O(n^{\frac{1}{2}-\frac{1}{2d}})$

$\boxed{\text{Sketch HW}}$ If H vcdim d & all edges have size $\geq \varepsilon . n$
$\exists$ hitting set $\leq O(\frac{1}{\varepsilon} . \log \frac{1}{\varepsilon} . d)$



perte $\sqrt{n}$
perte en densité $\frac{\sqrt{n}}{n}$ $\frac{1}{\sqrt{n}}$
puis $\frac{1}{\sqrt{\frac{n}{2}}}$ $\rightsquigarrow$ Arrêt à taille $O(\frac{1}{\varepsilon^2})$