Oriented coloring of 2-outerplanar graphs

Louis Esperet[*] and Pascal Ochem[†]
LaBRI UMR CNRS 5800, Université Bordeaux I,
33405 Talence Cedex
FRANCE.

19th January 2006

Abstract

A graph G is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar. The oriented chromatic number of an oriented graph H is defined as the minimum order of an oriented graph H' such that H has a homomorphism to H'. In this paper, we prove that 2-outerplanar graphs are 4-degenerate. We also show that oriented 2-outerplanar graphs have a homomorphism to the Paley tournament QR_{67}, which implies that their (strong) oriented chromatic number is at most 67.

Keywords: combinatorial problems, oriented coloring, 2-outerplanar graphs.

1 Introduction

Oriented graphs are directed graphs without opposite arcs. In other words an oriented graph is an orientation of an undirected graph, obtained by assigning to every edge one of the two possible orientations. If G is a graph, $V(G)$ denotes its vertex set, $E(G)$ denotes its set of edges. A homomorphism from an oriented graph G to an oriented graph H is a mapping φ from $V(G)$ to $V(H)$ which preserves the arcs, that is $(x,y) \in E(G) \Longrightarrow (\varphi(x), \varphi(y)) \in E(H)$. We say that H is a *target graph* of G if there exists a homomorphism from G to H. The oriented chromatic number $\chi_o(G)$ of an oriented graph G is defined as the minimum order of a target graph of G. The oriented chromatic number $\chi_o(G)$ of an undirected graph G is then defined as the maximum oriented chromatic number of its orientations. Nešetřil and Raspaud introduced in [4] the *strong oriented chromatic number* of an oriented graph G (denoted by $\chi_s(G)$), which definition differs from that of $\chi_o(G)$ by requiring that the target graph is an oriented Cayley graph. Upper bounds on the (strong) oriented chromatic number have been found for various subclasses of planar graphs. In particular:

1. if G is a planar graph, then $\chi_o(G) \leq 80$ [6].
2. if G is an outerplanar graph, then $\chi_s(G) \leq 7$ [7].

[*] *esperet@labri.fr*
[†] *ochem@labri.fr*

A graph G is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar. The second author proved that 2-outerplanar graphs have an acyclic partition into three independent sets and an outerplanar graph [5]. By Theorem 1 in [1], the oriented chromatic number of a 2-outerplanar graph is thus at most $2^{4-1} \times (1+1+1+7) = 80$. The same result follows from the bound of Raspaud and Sopena [6] holding for planar graphs.

In Section 2, we prove among other results that any 2-outerplanar graph G is 4-degenerate, *i.e.* every subgraph H of G has minimum degree at most 4. In Section 3, we use these results to show that 2-outerplanar graphs have a homomorphism to QR_{67}, which improves the previous bounds of 80.

2 Structural properties of 2-outerplanar graphs

Definition 1 *A 2-outerplanar graph embedded in the plane is said to be a* block *if its external face is an induced cycle.*

Theorem 1 *If G is a 2-outerplanar graph, then it contains a $^{\leq}4$-vertex.*

Proof. Let G be a 2-outerplanar graph embedded in the plane. We consider the subgraph H induced by the external face of G. H is an outerplanar graph, so it contains an internal face F incident to at most one other internal face of H (see Proof of Lemma 2 in [3]). Let B be the subgraph of G induced by the vertices of F and the vertices inside F. By construction, the graph B obtained is a block. Moreover, B contains only two vertices x and x' such that the degree of x and x' in G may be higher than their degree in B. By construction, x and x' are two adjacent vertices belonging to the external face of B (see Figure 1).

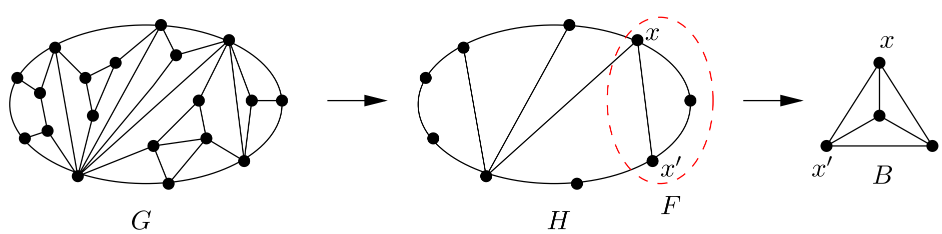

Figure 1: The decomposition of a 2-outerplanar graph into blocks.

Let B_c be the graph induced by the external face of B, and B_o be the graph obtained from B by removing the vertices of B_c. By definition of 2-outerplanar graphs, B_o is outerplanar. So it contains two non-adjacent 2-vertices u and v (see Figure 2).

As mentioned above, vertices of B_o have the same degree in B and in G, so $d_B(u) = d_G(u)$ and $d_B(v) = d_G(v)$. Let us find a $^{\leq}4$-vertex in B. If B_o contains a $^{\leq}4$-vertex, it is done. Else, it means that B_o contains only $^{\geq}5$-vertices; in particular u (resp. v) is adjacent to three vertices u_1, u_2, u_3 (resp. v_1, v_2, v_3), where $u_1u_2u_3$ (resp. $v_1v_2v_3$) is an induced P_3 of B_c (see Figure 3).

We now use the fact that B contains only two vertices x and x' having a degree in G possibly higher than their degree in B. As xx' is an edge of B_c, this means that u_2 or v_2 have

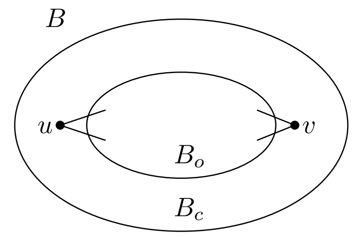

Figure 2: The decomposition of B into B_c and B_o.

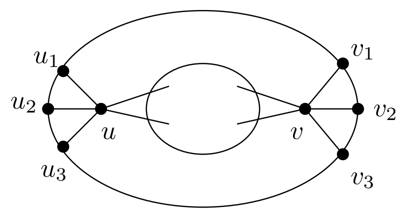

Figure 3: u and v have three neigbors in B_c.

the same degree in B and in G, i.e. $d_G(u_2) = d_B(u_2) = 3$ or $d_G(v_2) = d_B(v_2) = 3$. Hence B always contains a vertex with degree at most 4 in G. □

We now prove that outerplanar graphs have properties stronger than 2-degeneration, in order to find more precise configurations in 2-outerplanar graphs.

Lemma 1 *Let G be an outerplanar graph. G contains either a 1-vertex, two adjacent 2-vertices, a 2-vertex adjacent to a 3-vertex as depicted in Figure 4.a, or two 2-vertices adjacent to a 4-vertex as depicted in Figure 4.b.*

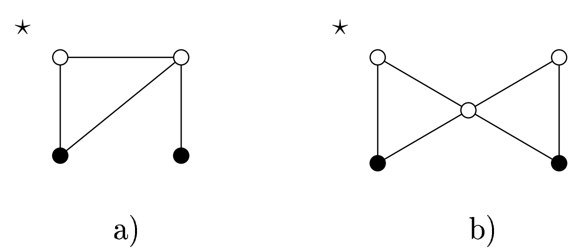

Figure 4: Unavoidable configurations in an outerplanar graph without two adjacent 2-vertices. The star symbol indicates the external face.

Proof. We prove this lemma by induction. Let G be an outerplanar graph, and let v be a 2-vertex of G (v exists, see [3] for details). The graph $H = G \backslash v$ is outerplanar, and smaller than G. By induction, H contains either two adjacent 2-vertices, or the configurations of Figure 4. If v is not adjacent to such a configuration of H, then it is a configuration of G, and the induction is finished. Else v is adjacent to a configuration, and we have to make the distinction between various cases. Notice that the neighbors of v must be adjacent in H in order to obtain an outerplanar graph.

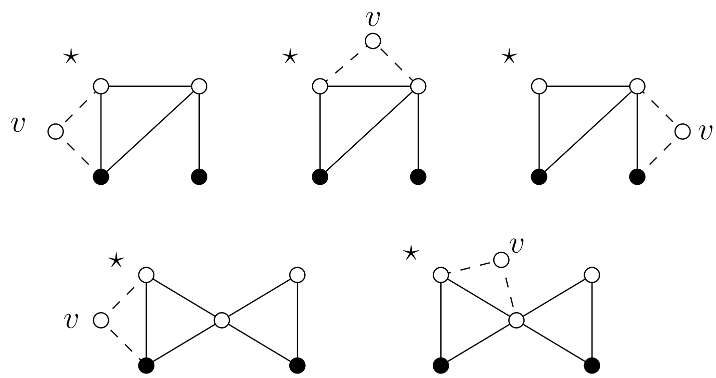

Figure 5: Induction step in the proof of Lemma 1.

- If H contains two adjacent 2-vertices, we obtain the configuration of Figure 4.a.
- If H contains a configuration of Figure 4, we obtain either the configuration of Figure 4.a, or the configuration of Figure 4.b (see Figure 5).

In any case, G contains one of the three configurations described earlier. □

We now use Lemma 1 to prove a key structural theorem on 2-outerplanar graphs admitting a block embedding in the plane. The following result can be extended to the whole class of 2-outerplanar graphs by using the same kind of proof as in Theorem 1.

Theorem 2 *Let G be a 2-outerplanar graph admitting a* block *embedding in the plane. G contains either a* $^{\leq}3$*-vertex, two adjacent 4-vertices, or the configuration depicted in Figure 6.*

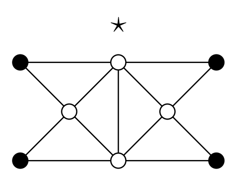

Figure 6: Unavoidable configuration in a 2-outerplanar block containing neither a $^{\leq}3$-vertex nor two adjacent 4-vertices.

Proof. We consider a block embedding of G in the plane. Then the subgraph induced by the external face is a cycle. Let G_c be this cycle and let G_o be the graph obtained from G by removing the vertices of G_c. By definition of G and G_c, the graph G_o is outerplanar. We then know by Lemma 1 that G_o contains either two adjacent 2-vertices, a 2-vertex having a neighbor of degree 3 as depicted in Figure 4.a, or two 2-vertices having a common neighbor of degree 4 as depicted in Figure 4.b.

- If G_o contains a 1-vertex or two adjacent 2-vertices, we easily find a $^{\leq}3$-vertex or two adjacent 4-vertices in G.
- If G_o contains a 2-vertex v adjacent to a 3-vertex u, we can prove that either $d_G(v) = 4$ or there is a vertex of degree 3 in G (which is a neighbor of v belonging to the external face). This is done by applying the same method as in the previous proof. Thus G must contain the configuration depicted in Figure 7. Notice that u and w are neigbors, else one of them would have degree at most 3. For reasons of planarity, if u is adjacent to another vertex of G_c, w cannot be adjacent to another vertex of G_o. Conversely, if w is adjacent to another vertex of G_o, u cannot be adjacent to a vertex of G_c. This proves that either u or w has degree 4 in G, say u. If there is no 3-vertex in G, we found two adjacent 4-vertices: u and v.

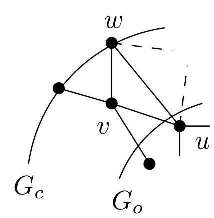

Figure 7: G_o contains a 2-vertex v adjacent to a 3-vertex u.

- If G_o contains two 2-vertices v and v' both adjacent to a 4-vertex u as depicted in Figure 4.b, we first prove that either v and v' have degree 4 in G, or G contains a 3-vertex (in which case the proof is finished). Let v_1 and v_2 (resp. v'_1 and v'_2) be the neigbors of v (resp. v') belonging to the external face. As depicted in Figure 8, we have to make a distinction between two cases : $\{v_1, v_2\}$ and $\{v'_1, v'_2\}$ are disjoint (case 1), or they have a vertex in common, say $v_2 = v'_1$ (case 2).

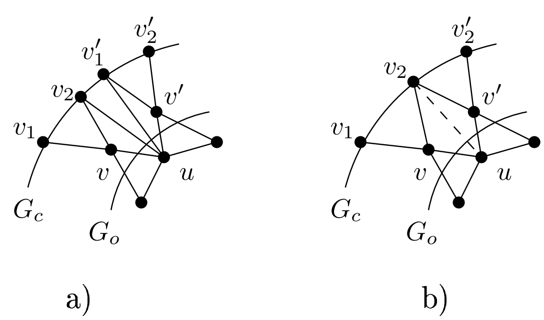

Figure 8: G_o contains two 2-vertices v and v' adjacent to a common 4-vertex u.

case 1 (see Figure 8.a) If v_2 and v'_1 have degree at least 4 in G, they both have to be adjacent to u, in which case $d_G(v_2) = d_G(v'_1) = 4$, and we found two adjacent 4-vertices in G.

case 2 (see Figure 8.b) If u is adjacent to $v_2 = v'_1$, we obtain exactly the configuration depicted in Figure 6. Otherwise, we simply have two adjacent 4-vertices (v and v_2).

□

3 Strong oriented coloring of 2-outerplanar graphs

Theorem 3 *If G is a 2-outerplanar graph, then $\chi_s(G) \leq 67$.*

For a prime power $q \equiv 3 \pmod 4$, the vertices of the Paley tournament QR_q are the elements of $\mathbb{F}_q$ and (i,j) is an arc in QR_q if and only if $j-i$ is a non-zero quadratic residue of $\mathbb{F}_q$. An *orientation vector* of size k is a sequence $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ in $\{0,1\}^k$. Let G be an oriented graph and $X = (x_1, x_2, \ldots, x_k)$ be a sequence of pairwise distinct vertices of G. A vertex y of G is said to be an *α-successor of* X if for every i, $1 \leq i \leq k$, we have $\alpha_i = 1 \Rightarrow (x_i, y) \in E(G)$ and $\alpha_i = 0 \Rightarrow (y, x_i) \in E(G)$. The graph G satisfies property $S_{k,n}$ if for every sequence $X = (s_1, s_2, \ldots, s_k)$ of k pairwise distinct vertices of G, and for every orientation vector α of size k, there exist at least n vertices in $V(G)$ which are α-successors of X.

A computer check proves the following lemma:

Lemma 2 *The tournament QR_{67} satisfies properties $S_{3,6}$ and $S_{4,1}$.*

We use the method of reducible configurations to show that every 2-outerplanar graph is QR_{67}-colorable. We define the partial order $\prec$ for the set of all graphs. Let $n_3(G)$ be the number of $^{\geq}3$-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if at least one of the following conditions hold:

- G_1 is a proper subgraph of G_2.
- $n_3(G_1) < n_3(G_2)$.

Note that this partial order is well-defined, since if G_1 is a proper subgraph of G_2, then $n_3(G_1) \leq n_3(G_2)$. So $\prec$ is a partial linear extension of the subgraph poset.

Let G be a 2-outerplanar graph having no homomorphism to QR_{67}, which is minimal with this property according to $\prec$.

Lemma 3 *G is 2-connected and does not contain a cut consisting in two adjacent vertices.*

Proof. If G is not 2-connected, then we can obtain a QR_{67}-coloring of G from the coloring of its 2-connected components, since QR_{67} is a circular tournament. Moreover G cannot contain a cut set consisting of two adjacent vertices, since QR_{67} is an arc-transitive tournament. □

Notice that Lemma 3 implies that every 2-outerplanar embedding of G is a block.

Lemma 4

1. *The graph G does not contain any $^{\leq}3$-vertex.*
2. *The graph G does not contain two adjacent 4-vertices.*
3. *The graph G does not contain the configuration depicted in Figure 6.*

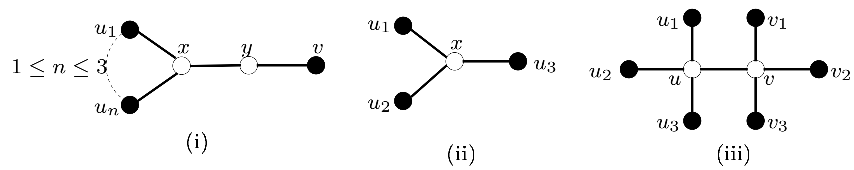

Figure 9: Forbidden configurations for Lemma 4.

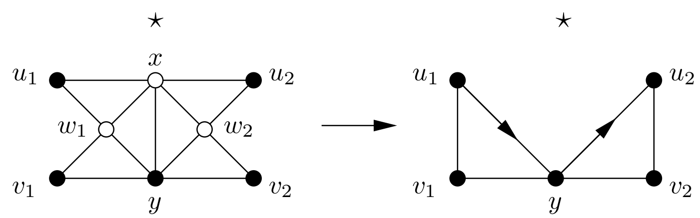

Figure 10: Construction of G' in the proof of Lemma 4.

Proof.

1. Consider configuration (i) in Figure 9. Let f be any QR_{67}-coloring of $G \setminus \{y\}$. By property $S_{3,6}$, we can choose f such that $f(x) \neq f(v)$ and extend this coloring to G. Consider now configuration (ii) in Figure 9. Notice that u_1, u_2, and u_3 are $^{\geq}3$-vertices, since configuration (i) with $n = 2$ is forbidden. Since QR_{67} is self-reverse, we assume w.l.o.g. that $d^-(x) \leq d^+(x)$ by considering either G or G^R. We have $d^-(x) \neq 0$, since otherwise we could extend any QR_{67}-coloring of $G \setminus \{x\}$ to G. Suppose now $d^-(x) = 1$, which is the only remaining case. Let us set $N^-(x) = \{u_1\}$, $N^+(x) = \{u_2, u_3\}$. We now consider the graph G' obtained from $G \setminus \{x\}$ by adding directed 2-paths joining respectively u_1 and u_2, and u_1 and u_3. Notice that if G is a block, then G' is a block. Moreover $G' \prec G$ since $n_3(G') = n_3(G) - 1$. Any QR_{67}-coloring f of G' induces a coloring of $G \setminus \{x\}$ such that $f(u_1) \neq f(u_2)$ and $f(u_1) \neq f(u_3)$, which can be extended to G.

2. Consider configuration (iii) in Figure 9. Let f be any QR_{67}-coloring of $G \setminus \{uv\}$ (that is we delete the edge uv). By property $S_{3,6}$, we can choose f such that $f(u) \notin \{f(v_1), f(v_2), f(v_3)\}$. Now by property $S_{4,1}$, we can choose f such that $f(v) \notin \{f(u), f(u_1), f(u_2), f(u_3)\}$ and extend this coloring to G.

3. Consider the configuration depicted in Figure 6. Let G' be the graph obtained from $G \setminus \{w_1, w_2, x\}$ by adding the arcs $\overrightarrow{u_1y}$ and $\overrightarrow{yu_2}$, and the arc $\overrightarrow{u_1v_1}$ (resp. $\overrightarrow{u_2v_2}$) if u_1 and v_1 (resp. u_2 and v_2) are not adjacent in G. This construction is depicted in Figure 10. Notice that if G is a block, then G' is a block. Moreover $G' \prec G$, since $n_3(G') = n_3(G) - 3$. Thus G' admits a QR_{67}-coloring which induces a QR_{67}-coloring f of $G \setminus \{w_1, w_2, x\}$ such that $f(u_1)$, $f(v_1)$, $f(y)$ (resp. $f(u_2)$, $f(v_2)$, $f(y)$; resp. $f(u_1)$, $f(u_2)$, $f(y)$) are pairwise distinct. By Property $S_{3,6}$, we can assign x a color $f(x) \notin \{f(u_1), f(u_2), f(y)\}$. By Property $S_{4,1}$, we can assign w_1 a color $f(w_1) \notin \{f(u_1), f(v_1), f(y), f(x)\}$ and assign w_2 a color $f(w_2) \notin \{f(u_2), f(v_2), f(y), f(x)\}$. We thus obtain a QR_{67}-coloring of G, which is a contradiction.

□

By Lemma 3 G is a block. Using Theorem 2, G must contain one of the configurations that are forbidden by Lemma 4. This contradiction completes the proof of Theorem 3.

References

[1] P. Boiron, E. Sopena, and L. Vignal. Acyclic improper colourings of graphs, *J. Graph Theory* **32** (1999), 97–107.

[2] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud, and E. Sopena. On the maximum average degree and the oriented chromatic number of a graph, *Discrete Math.* **206** (1999), 77–89.

[3] A. Hackmann and A. Kemnitz. List edge colorings of outerplanar graphs, *Ars Combinatoria* **60** (2001), 181–185.

[4] J. Nešetřil and A. Raspaud. Antisymmetric flows and strong colorings of oriented planar graphs, *Ann. Inst. Fourier* **49(3)** (1999), 1037–1056.

[5] P. Ochem. Ph.D. thesis, Université Bordeaux 1, 2005.
`http://dept-info.labri.fr/~ochem/these.ps`

[6] A. Raspaud and E. Sopena. Good and semi-strong colorings of oriented planar graphs, *Inform. Process. Lett.* **51(4)** (1994), 171–174.

[7] E. Sopena. Oriented graph coloring, *Discrete Math* **229(1–3)** (2001), 359–369.