ORCO – Graphs and Discrete Structures
October 5, 2022 – Lecture 2

1 Chromatic number and maximum degree

The *degree* of a vertex v (usually denoted by $d(v)$) in a graph G is the number of neighbors of v in G.
We now consider the following *greedy algorithm* to obtain a coloring of a graph G.
Order the vertices as $v_1, v_2, \ldots, v_n$. For $i = 1$ to n, color v_i with the smallest color (recall that colors are positive integers) that does not yet appear on its neighborhood.
Note that when choosing the color of a vertex v, at most $d(v)$ colors are forbidden to v and in particular, if v has at least $d(v) + 1$ available choices then it can always find a color that does not appear in its neighborhood. This shows the following.

Observation 1. *For any graph G with maximum degree Δ, the greedy algorithm finds a coloring with at most $\Delta + 1$ colors, and in particular $\chi(G) \leq \Delta + 1$.*

Note that this is best possible: odd cycles and complete graphs satisfy the bound above with equality. However, the following result (that we won't prove during the lecture) shows that these are the only extremal examples.

Theorem 2 (Brooks Theorem)**.** *If G is a connected graph with maximum degree at most Δ, distinct from an odd cycle or a complete graph, then $\chi(G) \leq \Delta$.*

We conclude with an exercise showing that the greedy algorithm can perform quite poorly for some instances.

Exercise 1. Consider the graph G_n with vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$, in which each vertex u_i is adjacent to all the vertices v_j with $j \neq i$ (see Figure 1 for a picture of G_4).
Show that $\chi(G_n) = 2$ for any $n \geq 2$. Show that there is an order on the vertices of G_n such that the greedy coloring in this order uses n colors.
Exercise 2. Show that for every graph G, there exists an ordering $v_1, \ldots, v_n$, on the vertices of G such that the greedy colouring algorithm performed on G with this order uses exactly $\chi(G)$ colors.

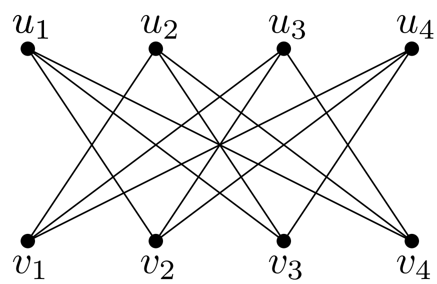

Figure 1: The graph G_4.

2 Degeneracy

We say that a graph G is k-*degenerate* (for some integer $k \geq 0$) if G has an ordering $v_1, \ldots, v_n$ of its vertices, such that for any i, the number of neighbors v_j of v_i with $j < i$ is at most k. In other words, v_i is of degree at most k in $G[\{v_1, \ldots, v_i\}]$.
The same proof as is the first lecture (for bounded degree graphs), shows that the greedy algorithm performs very well on k-degenerate graphs (using the same vertex ordering).

Observation 3. *If G is k-degenerate, then $\chi(G) \leq k + 1$.*

Homework – Find graphs for which equality holds, other than complete graphs and odd cycles.

In many applications we will need a slightly different (but equivalent) definition of k-degeneracy.
Before that, let us recall the notions of subgraphs and induced subgraphs. Given a graph $G = (V, E)$, we say that a graph $H = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. Moreover, we say that H is an *induced subgraph* of G if E' consists of all the edges of E with both endpoints in V'. You can think of a subgraph of G as a graph obtained from G by removing any set of vertices and edges. On the other hand, an induced subgraph of G is a graph obtained from G by only removing vertices (and the edges containing these vertices).

Theorem 4. *A graph G is k-degenerate if and only if each induced subgraph H of G contains a vertex of degree at most k in H.*

Proof. Assume first that G is k-degenerate. By definition, G has an ordering $v_1, \ldots, v_n$ of its vertices such that each vertex v_i has at most k neighbors v_j with $j < i$. Consider any induced subgraph H of G, and let S be the subset of $\{1, \ldots, n\}$ such that the vertices of H are precisely the vertices v_i with $i \in S$. Let ℓ be the maximum of S. Then v_ℓ has at most k neighbors v_j with $j < \ell$, and in particular v_ℓ has at most k neighbors in the set $\{v_i \,|\, i \in S\}$. It follows that v_ℓ has degree at most k in H, as desired.
Assume now that each induced subgraph H of G contains a vertex of degree at most k in H. In particular, G itself has a vertex of degree at most k (call it v_n). For $i = n-1, \ldots, 1$, we define v_i as a vertex of degree at most k in $G \setminus \{v_n, \ldots, v_{i+1}\}$ (such a vertex exists since any induced subgraph of G has a vertex of degree at most k). In this ordering, observe that for any i, the number of neighbors v_j of v_i with $j < i$ is precisely the degree of v_i in $G \setminus \{v_n, \ldots, v_{i+1}\}$, which is at most k by definition. It follows that G is k-degenerate, which concludes the proof. □

A class of graphs $\mathcal{C}$ is *hereditary* if it is closed under taking induced subgraphs (i.e. any induced subgraph of a graph of $\mathcal{C}$ is also in $\mathcal{C}$). The following simple corollary of Observation 3 and Theorem 4 has many applications.

Corollary 5. *Assume that $\mathcal{C}$ is a hereditary class such that any graph of $\mathcal{C}$ has a vertex of degree at most k. Then for any graph G of $\mathcal{C}$, $\chi(G) \leq k+1$.*

Proof. Let G be a graph of $\mathcal{C}$. Since any induced subgraph of G is in $\mathcal{C}$, any induced subgraph of G has a vertex of degree at mot k, and thus by Theorem 4, G is k-degenerate. It then follows from Observation 3 that $\chi(G) \leq k+1$, as desired. □