ORCO – Graphs and Discrete Structures
December 7, 2022 – Lecture 10

1 Complements of graphs

Given a graph G, the complement $\overline{G}$ of G is the graph with the same vertex-set as G, in which two vertices u and v are adjacent if and only if they are not adjacent in G.
Observe that the complement of the complement of G is G itself. Moreover, a clique in G is a stable set in $\overline{G}$, and stable set of G is a clique in $\overline{G}$. In particular $\alpha(G) = \omega(\overline{G})$ and $\omega(G) = \alpha(\overline{G})$.
Note also that if G has n vertices and m edges, then $\overline{G}$ has $\binom{n}{2} - m$ edges.

2 Excluding a clique

The following result of Turán is a precursor of the field of extremal graph theory (and extremal combinatorics in general).

Theorem 1 (Turán, 1941). *If a graph G on n vertices has no K_{r+1} as a subgraph, then G has at most $(1 - \frac{1}{r})\frac{n^2}{2}$ edges, and this is sharp.*

Proof. Let m be the number of edges of G. Since G has no K_{r+1}, we have $\omega(G) \le r$. By applying Theorem 5 from the previous lecture notes to the graph $\overline{G}$, we obtain that

$$r \ge \omega(G) = \alpha(\overline{G}) \ge \frac{n^2}{2(\binom{n}{2} - m) + n} = \frac{n^2}{n^2 - 2m} = \frac{1}{1 - 2m/n^2}.$$

It follows that $2m/n^2 \ge 1 - \frac{1}{r}$, and thus $m \le (1 - \frac{1}{r})\frac{n^2}{2}$, as desired.
We now observe that by taking r independent sets of size n/r, and adding all possible edges between different sets, we obtain a graph on n vertices with no K_{r+1}, and with $(1 - \frac{1}{r})\frac{n^2}{2}$ edges. This proves that the bound above is indeed sharp. □

A natural question is whether a similar result holds if we exclude an arbitrary subgraph H instead of K_{r+1}. The next theorem (which we will not prove here) shows that the answer only depends on the chromatic number of H.

Theorem 2 (Erdős-Stone, 1946)**.** *For any graph H, if a graph G on n vertices does not contain H as a subgraph, then G has at most $(1 - \frac{1}{\chi(H)-1})\frac{n^2}{2} + o(n^2)$ edges.*

Since $\chi(K_{r+1}) = r + 1$, this immediately implies the previous theorem (with some additional lower order terms).
Observe that when $\chi(H) = 2$ (i.e. when H is bipartite), this theorem implies that any graph on n vertices excluding H as a subgraph has $o(n^2)$ edges. In the next section, we consider the case where H is a 4-cycle (which is clearly bipartite), and we find the right order of magnitude for the number of edges in graphs excluding H.

3 Excluding a 4-cycle

Theorem 3. *If a graph G on n vertices has no 4-cycle, then G has at most $\frac{1}{2}n^{3/2}$ edges.*

Proof. Let m be the number of edges of G. A *cherry* in G is a pair $(v, \{u, v\})$ where u, v, w are vertices of G and $\{u, v\}$ is an unordered pair of neighbors of v in G. If we fix v there are $\binom{d(v)}{2}$ pairs $\{u, w\}$ such that $(v, \{u, v\})$ is a cherry, so $K = \sum_v \binom{d(v)}{2}$. On the other hand, if we fix $\{u, v\}$ there is at most one vertex v such that $(v, \{u, v\})$ is a cherry (if there was a second vertex with this property, then the graph would contain a 4-cycle). This shows that $\sum_v \binom{d(v)}{2} = K \leq \binom{n}{2}$. For the sake of exposition, let us replace this inequality by the slightly simpler

$$\sum_v \frac{d(v)^2}{2} \leq \frac{n^2}{2},$$

(which will lead to the same result, up to lower order terms). By convexity of the function $x \mapsto x^2$ (or by applying Cauchy-Schwarz inequality), it follows that $\sum_v d(v)^2 \geq \frac{1}{n}(\sum_v d(v))^2 = \frac{1}{n}(2m)^2$. As a consequence, $(2m)^2 \leq n^3$ and thus $m \leq \frac{1}{2}n^{3/2}$. □

We now construct examples showing that the bound given in the previous theorem is close from best possible.
A *projective plane of order q* is a set of elements (called *points*), together with a collection of sets of points (called *lines*) with the following properties.

1. there are $q^2 + q + 1$ points and $q^2 + q + 1$ lines

2. each point is in $q+1$ lines and each line contains $q+1$ points
3. for any pair of point there is a unique line containing them, and any pair of lines intersect in a unique point.

It is known that projective planes of order q exist whenever q is a prime power. Proving that they exist (or don't exist) for other values of q is an important open problem. However, by Bertrand's postulate, there is a prime number between n and $2n$ for any integer n so the projective planes of prime order are sufficient for most applications.

Given a projective plane of order q, we can construct a bipartite graph H_q as follows: H_q is a bipartite graph with bipartition (P, L) (P the points and L the lines), with an edge between $p \in P$ and $\ell \in L$ if and only if $p \in \ell$ in the projective plane.
Note that the resulting bipartite graph has no 4-cycle (otherwise two points would be contained in two lines, contradicting property 3 above). Moreover, the bipartite graph H_q has $n = 2q^2 + 2q + 2 \approx 2q^2$ vertices and $m = (q+1)(q^2+q+1) \approx q^3$ edges (since each vertex has degree $q+1$, by property 2 above). It follows that $m \approx (n/2)^{3/2}$ which is very close from the bound of Theorem 3.

4 Application of the construction of H_q

In Lecture 1 we saw how to construct *triangle-free* graphs of arbitrarily large chromatic number. Such constructions typically have size exponential in the chromatic number, so they are completely unusable in practice. We will see how to use the bipartite graph H_q constructed in the previous section to obtain triangle-free graphs of large chromatic number, where the number of vertices is at most polynomial in the chromatic number (with $|V(G)| = O(\chi(G)^3)$ more precisely). This is the best known deterministic construction for this problem (all the improvements use random graphs).

Let G be the graph constructed from H_q as follows: the vertices of G are the m edges of H_q, arbitrary ordered as $e_1, \ldots, e_m$. Then we add an edge between e_i and e_j whenever $i < j$ and there is an edge in H_q that connects the endpoint of e_i lying in L to the endpoint of e_j lying in P.

We first claim that G is triangle-free. To see this, consider three edges of H_q corresponding to a triangle in G. Then by definition of G we can find a C_4 in H_q, which is contradiction (see the previous section).

Next, we claim that $\alpha(G) \leq 2(q^2+q+1)$. To see this, consider any independent S in G and the corresponding set T of edges in H_q. We claim that the subgraph of H_q formed by the edges of T is acyclic: if not, consider a cycle and look at the edge e_i of lowest index on this cycle. Following the next two edges on the cycle, we see that the vertex corresponding to e_i in G must be adjacent to the vertex corresponding to another edge of the cycle, which contradicts the fact that S is an independent set. As T forms an acyclic graph in H_q, which has at most $n = 2(q^2+q+1)$ vertices, $|T| \leq n-1 \leq 2(q^2+q+1)$, as desired.

We now recall that for any graph G on N vertices, $\chi(G) \geq N/\alpha(G)$ (see Lecture 9). So here $N = m \approx q^3$ while $\alpha(G) \leq 2(q^2+q+1) \approx 2q^2$, so $\chi(G) \geq q/2$.