**ORCO – Graphs and Discrete Structures**
**October 12, 2022 – Lecture 3**

# 1 Planar graphs

A graph is *planar* if it has an embedding in the plane with no edge-crossings. The connected component of the plane minus the embedding are called the *faces*. The *degree of a face* is the number of edges on a boundary walk (counted with multiplicity). A fundamental result about planar graphs is the following.

**Theorem 1** (Euler's Formula)**.** *If $G$ is a connected planar graph, embedded in the plane, with $n$ vertices, $m$ edges, and $f$ faces, then $n - m + f = 2$.*

Note that it shows in particular that the number of faces of a planar graph does not depend on the embedding the graph (and thus we can remove "embedded in the plane" in the theorem above).
We will not prove Euler's formula formally (during the lectures we have seen two sketch of proofs, one using discharging method).

We will now deduce the following simple result from Euler's Formula.

**Lemma 2.** *Any planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges.*

*Proof.* We can assume that the graph is connected (otherwise we consider each connected component separately). Let $m$ be the number of edges and $f$ be the number of faces of $G$. By Euler's Formula, we have $n - m + f = 2$ and thus $f = 2 - n + m$. A simple counting argument shows that the sum of the degrees (number of edges in a boundary walk) of the faces of $G$ is equal to $2m$, and since each face has degree at least 3, we have $2m \geq 3f$ and thus $f \leq \frac{2}{3}m$. It follows that $2 - n + m \leq \frac{2}{3}m$ and thus $m \leq 3n - 6$, as desired. $\square$

*Homework – Similarly, find a bound on the number of edges of a triangle-free planar graph.*

Recall that the sum of the degrees of the vertices of a graph is precisely twice the number of edges of that graph. Thus, it follows from Lemma 2 that any planar graph has average degree less than 6. In particular:

**Corollary 3.** *Any non-empty planar graph has a vertex of degree at most 5.*

*Homework – Similarly, find a bound on the minimum degree of a triangle-free planar graph.*

# 2 Coloring planar graphs

Since any induced subgraph of a planar graph is planar, the class of planar graphs is hereditary, and thus Corollary 3 implies that planar graphs are 5-degenerate (see last week lecture). A direct consequence of this, using another result proved last week, is the following simple result.

**Corollary 4.** *Any planar graph has chromatic number at most 6.*

Perhaps the most important result in graph coloring is the following improved version, solving a problem posed by Guthrie in 1852.

**Theorem 5** (The 4 Color Theorem)**.** *Any planar graph has chromatic number at most 4.*

We will not prove it here, but we will prove the following version instead (that goes halfways between Corollary 4 and Theorem 5.

**Theorem 6** (The 5 Color Theorem)**.** *Any planar graph $G$ has chromatic number at most 5.*

*Proof.* We prove the theorem by induction on the number of vertices. The conclusion is clear if $G$ has at most 1 vertex, so assume $G$ has at least 2 vertices. Consider any fixed embedding of $G$ in the plane (without edge-crossings). The first case is that $G$ contains a vertex $v$ of degree at most 4. Then we can color $G - v$ by induction with at most 5 colors, and since $v$ has degree at most 4, we can extend the coloring to $v$ ($v$ has 5 choices and at most 4 are forbidden). Since every planar graph contains a vertex of degree at most 5, the last case is that $G$ contains a vertex $v$ of degree precisely 5. Again, we color $G - v$ by induction and attempt to extend this coloring to $v$. If some color (among $1, \ldots, 5$) does not appear among the neighbors of $v$, we can extend the coloring to $v$. Otherwise, we can assume by symmetry that the neighbors $v_1, \ldots, v_5$ (in clockwise order around $v$) are colored such that $v_i$ has color $i$ for any $1 \leq i \leq 5$. Let $G_{13}$ be the set of vertices of $G$ colored 1 or 3, and let $C_{13}$ be the connected component of $G_{13}$ containing $v_1$. In $C_{13}$, interchange colors 1 and 3 (i.e. recolor all vertices colored 1 with color 3, and all vertices colored 3 with color 1). The resulting coloring is still a 5-coloring of $G - v$, and if $v_3 \notin C_{13}$, color 1 does not appear in the neighborhood of $v$ and we can extend the coloring to $v$. Assume now that $v_3 \in C_{13}$ (which implies that there is a path $P_{13}$ of vertices colored 1 or 3 between $v_1$ and $v_3$ in $G-v$). Now do exactly the same thing with $v_2$ and $v_4$. Again, we can remove color 2 from the neighborhood of $v$ unless there is a path $P_{24}$ colored 2 or 4 between $v_2$ and $v_4$ in $G-v$. But since $v_1, \dots, v_5$ are in clockwise order around $v$, the paths $P_{13}$ and $P_{24}$ must intersect. As they are vertex disjoint, two edges of $P_{13}$ and $P_{24}$ must cross, which contradicts the fact that the embedding of $G$ was planar. $\square$

# 3 Planar graph drawing

The goal of this section is to prove Fáry's theorem (originally due to Wagner, 1936), which states that for any planar drawing of a planar graph there is an equivalent planar drawing in which all edges are straight-line segments.

We will first need a couple of ingredients. The first is a simple observation that can be deduced from the fact that any $n$-vertex planar has at most $3n-6$ edges (assuming $n \geq 3$), proved in the last lecture.

**Lemma 7.** *Every planar graph $G$ on $n \geq 4$ vertices has at least 4 vertices of degree at most 5.*

*Proof.* Note that if some graph obtained from $G$ by adding edges has at least 4 vertices of degree at most 5, then $G$ also has at least 4 vertices of degree at most 5. Hence, we can assume without loss of generality that $G$ is edge-maximal with respect to being planar, and in particular $G$ has minimum degree at least 3. Assume for the sake of contradiction that $G$ has at most 3 vertices of degree at most 5. So $n-3$ vertices have degree at least 6, and the remaining 3 have degree at least 3. It follows that the sum of degrees in $G$ is at least $6(n-3)+3\cdot 3 = 6n-9$, and thus $G$ has at least $3n-4$ edges, a contradiction. $\square$