ORCO – Graphs and Discrete Structures
December 14, 2022 – Lecture 11

1 Ramsey's theorem

For two integers $k, \ell \geq 2$, the *Ramsey number* $R(k, \ell)$ is defined as the minimum integer n such that in any coloring of the edges of K_n with colors blue and red, there is a K_k with all edges colored blue or a K_ℓ with all edges colored red.
It is not difficult to check that for any k, $R(k, 2) = R(2, k) = k$. The first interesting case is $R(3, 3)$.

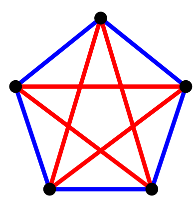

Figure 1: A coloring of the edges of K_5 showing that $R(3, 3) \geq 6$.

Theorem 1. $R(3, 3) = 6$.

Proof. The figure above shows a coloring of K_5 without red triangles and blue triangles, so $R(3, 3) \geq 6$. Now assume that the edges of K_6 are colored with colors red and blue, and consider an arbitrary vertex v of K_6. Since v has degree 5, it must be adjacent to 3 edges of the same color, say blue by symmetry. The 3 corresponding neighbors x, y, z of v form a triangle. If some edge of this triangle is blue, then together with v this forms a blue triangle. Otherwise all the edges of the triangle are red, and we have a red triangle. It follows that $R(3, 3) \leq 6$, and thus $R(3, 3) = 6$. □

With a similar approach we can prove the following general result, that shows in particular that $R(k, \ell)$ is finite for any k and ℓ.

Theorem 2 (Ramsey, 1930)**.** *For any $k, \ell \geq 3$, $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$. In particular, $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$ and thus $R(k, k) \leq 2^{2k}$.*

Proof. Consider a vertex v in the complete graph on $N = R(k-1,\ell) + R(k,\ell-1)$ vertices, with an arbitrary coloring of its edges with colors blue and red. Then v is incident to at least $R(k-1,\ell)$ blue edges, or incident to at least $R(k,\ell)$ red edges (since otherwise it would be incident to at most $R(k-1,\ell)-1+R(k,\ell-1)-1 = N-2$ edges, a contradiction). In the first case, the graph induced by the endpoints of the $R(k-1,\ell)$ blue edges must contain a blue K_{k-1} (and thus our K_N contains a blue K_k, if we add v), or a red K_ℓ, as desired. In the second case, the graph induced by the endpoints of the $R(k,\ell-1)$ red edges must contain a blue K_k, or a red $K_{\ell-1}$ (and thus our K_N contains a red K_ℓ, if we add v), as desired.
The second part of the statement follows easily by induction from $R(k,2) = R(2,k) = k$ and the standard recurrence relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. □

We obtain the following important corollary.

Corollary 3. *Every graph on n vertices contains a clique of size $\frac{1}{2}\log n$ or a stable set of size $\frac{1}{2}\log n$.*

Proof. Given a graph G on n vertices, color the edges of G with color blue, and add red edges between any pair of non-adjacent vertices in G. We obtain a 2-coloring of the edges of K_n, the complete graph on n vertices, with colors red and blue. Setting $k = \frac{1}{2}\log n$, we have $n = 2^{2k}$, and thus $n \geq R(k,k)$ (by Theorem 2). It follows that we can find a blue clique on k vertices (and thus a clique on k vertices in G), or a red clique on k vertices (which is exactly a stable set on k vertices in G). □

Similarly, last week's construction of a triangle-free graph H_q with small independence number (see Section 4 in Lecture 10) directly implies the following.

Corollary 4 (of the construction of H_q last week)**.** *For any ℓ, $R(3,\ell) \geq (\ell/2)^{3/2}$.*

Actually, better estimates are known for $R(3,\ell)$, which is of order $\ell^2/\log\ell$, but the bound above is the best known bound that follows from a deterministic construction.

We will now see that the bound $\frac{1}{2}\log n$ in Corollary 3 is close to best possible. This is one of the earliest applications of the so-called *Probabilistic method*, pioneered by Paul Erdős. Recall that the *random graph* $G(n,p)$ is the graph with n vertices in which for any two distinct vertices u,v we add an edge between u and v at random with probability p, independently of the other pairs of vertices.

Theorem 5 (Erdős, 1947)**.** *The random graph $G(n, \frac{1}{2})$ contains no clique of size $2\log n$ and no stable set of size $2\log n$, with probability tending to 1 as $n \to \infty$.*

Proof. The probability that a given set S of s vertices is an independent set is $(1/2)^{\binom{s}{2}}$, and similarly for the probability that S induces a clique. So the expectation of the number of independent sets of size s (and similarly, of cliques of size s) is

$$\binom{n}{s} \cdot (1/2)^{\binom{s}{2}} \leq \frac{n^s}{s!} \cdot \frac{2^{s/2}}{2^{s^2/2}}.$$

For $s = 2\log n$ we have $n^s = 2^{s\log n} = 2^{s^2/2}$ and thus this expectation is at most $\frac{2^{s/2}}{s!}$, which tends to 0 as $n \to \infty$. So the probability that $G(n, \frac{1}{2})$ contains a clique or independent set of size at least $s = \frac{1}{2}\log n$ tends to 0 as $n \to \infty$. □

Note that in the outcome of $G(n, \frac{1}{2})$, each (labelled) n-vertex graph appears with the same probability. So Theorem 5 can be rephrased by saying that as $n \to \infty$, the proportion of n-vertex graphs no clique of size $2\log n$ and no stable set of size $2\log n$ tends to 1. This is sometimes written as: *almost all n-vertex graphs have no clique of size $2\log n$ and no stable set of size $2\log n$.*

Theorem 2 is a "baby" version of a much more general statement proved by Ramsey. In the statement of Theorem 2 we only had 2 colors (red and blue), and we were coloring the edges of the complete graph, or equivalently the pairs of vertices, or 2-element subsets of $[n] = \{1, \dots, n\}$. In the more general statement we will allow r colors, and we will color the s-element subsets of $[n]$ with these r colors.

Theorem 6 (Ramsey, 1930)**.** *For every integers r, s, and $t \geq s$, there is an integer $n = n(r, s, t)$ such that in any coloring of the s-element subsets of $[n]$, there is a subset $X \subseteq [n]$ of size at least t, such that all s-element subsets of X are colored with the same color.*

Note that Theorem 2 corresponds to the case $r = 2$ and $s = 2$ of Theorem 6 (and we indeed have $n = R(t, t)$). It should be noted that while in Theorem 2 the depends between n and k was (only) exponential, in Theorem 6 the dependence in n and the parameters r, s, t is absolutely huge.

2 Applications

We will now see two quick applications of Ramsey's theorem in number theory and geometry. In both cases, these results can be proved without using Ramsey's theorem, with much better bounds, but the proofs are usually more complicated.

Theorem 7 (Schur, 1916)**.** *For any integer r, there is an integer n such that in any r-coloring of the elements of $[n]$, there are three distinct elements x, y, z with the same color such that $z = x + y$.*

Proof. Let n be given by Theorem 6 with r colors, $s = 2$ and $t = 3$. Consider any r-coloring c of $[n]$, and assign to each pair $\{i, j\}$ the color $c(|j - i|)$. By Theorem 6, there are 3 elements $i < j < k$ such that $c(j - i) = c(k - j) = c(k - i)$. It then suffices to observe that $(j - i) + (k - j) = (k - i)$. So we have three elements $x = j - i$, $y = k - j$, and $z = k - i$ such that x, y, z have the same color, and $z = x + y$, as desired. □

We say that a set X of points in the plane is in *general position* if no 3 points of X are collinear. A set Y of points is in *convex position* if no point of Y is contained in the convex hull of the other points of Y (equivalently, the points of Y are the vertices of a convex polygon).

Theorem 8 (Erdős and Szekeres, 1935)**.** *For any integer k, there is an integer n such that for any set X of n points in the plane in general position, there is a subset $Y \subseteq X$ in convex position with $|Y| = k$.*

Proof. Let n be given by Theorem 6 with 2 colors, $s = 3$ and $t = k$. Assign to each triple x, y, z of points from X the color $c(\{x, y, z\}) = 0$ if the number of points of X in the interior of the triangle xyz is even, and $c(\{x, y, z\}) = 1$ otherwise. By Theorem 6, there is a set Y of k points, such that all triples of points from Y have the same color. We claim that the points of Y are in general position. If not, then there must be 4 points $x_0, x_1, x_2, x_3 \in Y$ such that x_0 is in the interior of the triangle $x_1x_2x_3$. Since the points of X are in general position, all points in the interior of $x_1x_2x_3$ distinct from x_0 are in the interior of $x_0x_1x_2$, $x_0x_2x_3$, or $x_0x_3x_1$, so it follows that $c(\{x_1, x_2, x_3\}) \equiv c(\{x_0, x_1, x_2\}) + c(\{x_0, x_2, x_3\}) + c(\{x_0, x_3, x_1\}) + 1 \pmod 2$. This clearly contradicts the fact that $c(\{x_1, x_2, x_3\}) = c(\{x_0, x_1, x_2\}) = c(\{x_0, x_2, x_3\}) = c(\{x_0, x_3, x_1\})$. □