ORCO – Graphs and Discrete Structures
October 19, 2022 – Lecture 4

1 Planar graph drawing (resuming)

A planar graph is said to be *outerplanar* if it has a planar drawing in which all vertices lie on the outerface.
The *dual* of (some planar drawing of) G is the multigraph whose vertices are the faces of G, and whose edges are in bijection with the edges of G : for each $e \in E(G)$, let f and f' be the faces appearing on each side on e, then ff' is an edge in the dual of G.
The *weak dual* of (some planar drawing of) G is the graph whose vertices are the internal faces of G, and such that two vertices are adjacent if and only if the corresponding faces of G share an edge.

Lemma 1. *Every outerplanar graph G contains a vertex of degree 2.*

Proof. We can again assume that G is edge-maximal with respect to being outerplanar graph. In particular, G has no cut-vertices, and if we consider any planar drawing of G in which all vertices lie on the outerface, the outerface is a cycle (without repeated vertices), and all internal faces are triangles. The *weak dual* of (some planar drawing of) G is the graph whose vertices are the internal faces of G, and such that two vertices are adjacent if and only if the corresponding faces of G share an edge. It is not difficult to check that the weak dual of an edge-maximal outerplanar graph is a tree (if it contains a cycle, then some vertex of G does not lie on the outerface). Note that every tree has a vertex of degree 1, and the corresponding triangular face of G must share two edges with the outerface. It follows that this face contains a vertex of degree 2, as desired. □

Since outerplanar graphs form a hereditary family, this shows that outerplanar graphs are 2-degenerate. We have the following immediate corollary.

Corollary 2. *Every outerplanar graph is 3-colorable.*

In the *Art gallery problem* we are given an n-vertex polygon in the plane (an n-gon, in short), and the goal is to place guards inside the n-gon so that all the (inside) region bounded by the n-gon is under the surveillance of some guard (a point is under the surveillance of a guard if the segment connecting them is contained in the region bounded by the n-gon).

Theorem 3. $\lfloor n/3 \rfloor$ *guards are sufficient to guard an n-gon.*

Proof. Arbitrarily add edges to the internal faces of the n-gon until all internal faces are triangles. We obtain an n-vertex outerplanar graph G, which is 3-colorable by Corollary 2, and thus some color $i \in \{1, 2, 3\}$ is such that at most $\lfloor n/3 \rfloor$ vertices of G are colored i. Note that all triangles of G are colored $1, 2, 3$, and thus if we place the guards at the location of the vertices colored i, all the region bounded by the n-gon is guarded. □

We will only need the case $n \geq 5$ of Theorem 3, which can be rephrased as follows.

Corollary 4. *Any n-gon with $3 \leq n \leq 5$ has an interior point x, such that each segment connecting x to the vertices of the n-gon is contained in the region bounded by the n-gon.*

We are now ready to prove Fáry's theorem.

Theorem 5. *For every planar embedding of a planar graph G, there is an equivalent embedding in which all edges are straight-line segments.*

Proof. As before, we can assume without loss of generality that G is edge-maximal with respect to the property of being planar (if we can draw a supergraph of G with edges as straight-line segments, then G certainly also has such a drawing). In particular, the outerface is a triangle, and for every vertex v, there is a circular order on the neighbors of v such that any two consecutive neighbors in this order are adjacent. It follows that if we remove v, then these neighbors form a face in the corresponding embedding of $G \setminus \{v\}$. Let us prove the result by induction on the number n of vertices. This is certainly true if $n \leq 3$ (in this we only have a single vertex, a single edge, or a triangle), so assume that $n \geq 4$. Since the outerface is a triangle (containing 3 vertices), and G has at least 4 vertices of degree at most 5 (see a Lemma from last week), it follows that G contains a vertex v of degree at most 5 that does not lie on the outerface of G. We remove v and apply induction on the resulting embedding of $G \setminus \{v\}$. We obtain a straight-line drawing of $G \setminus \{v\}$ in which the neighbors of v form a face (and thus a k-gon, for $3 \leq k \leq 5$). It follows from Corollary 4 that we can add v inside the region bounded by the k-gon, and connect v to its neighbors with straight-line segments without creating crossings. □

2 Interval graphs

Recall that for any graph G, $\omega(G) \leq \chi(G)$. In this lecture we are going to see natural graph classes for which equality holds.

A graph G with vertices $v_1, \ldots, v_n$ is said to be an *interval graph* if there exist intervals $I_1, \ldots, I_n$, such that any two vertices v_i and v_j are adjacent in G if and only if the corresponding intervals I_i and I_j intersect. The interval $I_1, \ldots, I_n$ are called *the interval representation* of G.

In such a graph G, consider the interval I_i with rightmost left end. Note that by definition, all the neighbors of the corresponding vertex v_i also intersect this left end, and thus v_i, together with its neighbors, forms a clique.

Observation 6. *Any interval graph G contains a vertex v whose neighborhood is a clique. In particular, v has degree at most $\omega(G) - 1$.*

Note that each induced subgraph of an interval graph is also an interval graph (starting with an interval representation of the graph and removing some intervals, we can get interval representations of any induced subgraph of G). Applying Corollary 3 of the Lecture notes of the 2nd lecture, we obtain:

Theorem 7. *Any interval graph G has chromatic number $\omega(G)$.*

Another (equivalent) way to look at this theorem (perhaps more algorithmic) is the following.

Given an interval graph G with interval representation $I_1, \ldots, I_n$, sort the interval by their left end and color them using the greedy algorithm from left to right. Then the observations above shows that the resulting coloring is a coloring with $\omega(G)$ colors.

3 Chordal graphs

An equivalent way to define interval graphs would be to say that the interval $I_1, \ldots, I_n$ are subpaths of a given path. This yields the following natural generalisation.

A graph G with vertices $v_1, \ldots, v_n$ is a *chordal graph* if there is a tree $\mathcal{T}$ and subtrees $T_1, \ldots, T_n$ of $\mathcal{T}$ such that for any i, j, v_i and v_j are adjacent in G if and only if T_i and T_j intersect.

While this definition is quite natural, it is usually easier to work with the following equivalent definition.

Given a graph G, a *subtree representation* of G is a tree $\mathcal{T}$ together with sets $(B_t)_{t\in\mathcal{T}}$ of vertices of G (the set B_t is called the *bag* of t), with the following properties.

1. u and v are adjacent in G if and only if there is a bag B_t containing u and v in the subtree representation, and

2. for any vertex v, the set of nodes t of $\mathcal{T}$ whose bag B_t contains v forms a subtree of $\mathcal{T}$ (equivalently, if v is in two bags B_t and B_s, then v lies in all the bags $B_{t'}$ such that t' is on the path between t and s in $\mathcal{T}$).

Now it can be seen that a graph is a chordal graph if and only if it has a subtree representation. To see why this holds, if you start with some subtrees $T_1, \ldots, T_n$ of a tree $\mathcal{T}$ as above, then for any node t of $\mathcal{T}$ you can define B_t as the set of vertices v_i such that the corresponding subtree T_i contains t (it is not difficult to check that the definition of a subtree representation is then satisfied). On the other hand, starting with a subtree representation $\mathcal{T}$ with bags $(B_t)_{t\in\mathcal{T}}$, we can define the subtree T_i of a vertex v_i as the set of nodes t of $\mathcal{T}$ such that B_t contains v_i. In this case condition 1 in the definition above implies that two vertices are adjacent if and only if the corresponding subtrees intersect.

It follows directly from the definition that any interval graph is a chordal graph (and in fact, the interval graphs are precisely the chordal graphs with a subtree representation that is a path), but a natural question is whether there are chordal graphs that are not interval graphs (that is, we have really defined a more general class of graphs).
We start by proving the following.

Proposition 8. *Trees are chordal graphs.*

Proof. We prove the result by induction of the number of vertices of a tree T. The result is clear if T contains a single vertex, so assume it contains at least two vertices. Let u be a leaf of T, and let v be its unique neighbor in T. By induction, find a subtree representation of $T \setminus \{u\}$, say with underlying tree $\mathcal{T}$ and bags $(B_t)_{t\in\mathcal{T}}$. Consider a node $t \in \mathcal{T}$ such that $v \in B_t$, and add a new leaf t^* in $\mathcal{T}$ whose unique neighbor is t. Define $B_{t^*} = \{u, v\}$. We now have a subtree representation of T, as desired. $\square$

It can be checked that the graph of Figure 1 is not an interval graph. However it is a tree, and it is thus chordal by Proposition 8.

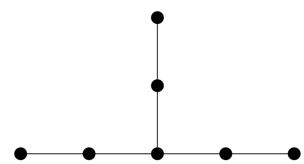

Figure 1: A chordal graph which is not an interval graph.

Given a chordal graph G, let us say that a subtree representation $\mathcal{T}, (B_t)_{t\in\mathcal{T}}$ of G is *pruned* if for any leaf t of $\mathcal{T}$, the bag B_t of t contains a vertex of G that does not appear in any other bag.

Proposition 9. *Any chordal graph has a pruned subtree representation.*

Proof. Assume some leaf t is such that all the vertices of B_t appear in another bag. Then by connectivity, all the vertices of B_t also appear in the bag B_s of the unique neighbor s of t in $\mathcal{T}$. In this case we can simply delete t from $\mathcal{T}$ and the graph it describes remains unchanged. If we repeat this procedure until no such leaf t exists we have the property that each leaf contains a vertex that does not appear in another bag, as desired. □

Pruned subtree representations are very useful. Let us now prove the following generalisation of Observation 6.

Proposition 10. *Any chordal graph G contains a vertex v whose neighborhood is a clique. In particular, v has degree at most $\omega(G) - 1$.*

Proof. Consider a pruned subtree representation $\mathcal{T}$ of G, with bags $(B_t)_{t\in\mathcal{T}}$, and a leaf t of $\mathcal{T}$. By the definition of a pruned subtree representation, B_t contains a vertex v that does not appear in another bag. Since for any neighbor u of v, u and v have to be in a common bag, it follows that B_t contains all the neighbors of v, and thus v and its neighbors forms a clique. □

As in the case of interval graphs, observe that each induced subgraph of a chordal graph is also a chordal graph (starting with a subtree representation of the graph and removing some subtrees, we can get subtree representations of any induced subgraph of G). Applying Corollary 4 of the Lecture notes of the 2nd lecture, we obtain:

Theorem 11. *Any chordal graph G has chromatic number $\omega(G)$.*

4 Perfect graph

A graph G is perfect if and only if any induced subgraph H of G satisfies $\chi(H) = \omega(H)$.

What we have proved in the previous section is that chordal graphs (and interval graphs) are perfect.