ORCO – Graphs and Discrete Structures
September 29, 2021 – Lecture 1

1 Definitions

Given a graph G and an integer k, a *k-coloring* of G is an assignment of k colors (usually denoted by $\{1, \dots, k\}$) to the vertices of G such that any two adjacent vertices have different colors. Given a coloring, the set of all vertices with a given color is usually called *a color class* (an a coloring can be thought of as a partition of the vertex set into color classes).
The *chromatic number of* G, denoted by $\chi(G)$, is the least k such that G has a k-coloring.

If a graph G has a k-coloring we also say that G is *k-colorable*, and if $\chi(G) = k$ we also say that G is *k-chromatic*.

A *clique* in a graph G is a set of pairwise adjacent vertices in G. The *clique number* of G, denoted by $\omega(G)$, is the maximum number of vertices in a clique of G.
Since in any coloring of G, all the vertices of a clique must have distinct colors, we have the following simple observation.

Observation 1. *For any graph G, $\omega(G) \leq \chi(G)$.*

It is easy to see that there exist graphs for which the inequality above is strict (for instance, odd cycles on at least 5 vertices). In the next section, we show how to construct graphs for which the difference between the chromatic and cliques numbers is arbitrarily large.

Before that, let us study the class of 2-colorable graphs, also known as *bipartite graphs*. A 2-coloring is also called a *bipartition*. A classical result in graph theory is the following.

Theorem 2. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. Since odd cycles are 3-chromatic, any graph that contains an odd cycle has chromatic number at least 3, which proves the first direction. To prove the second direction, consider a graph G with no odd cycle. We can assume that G is connected (otherwise we consider each connected component separately). Fix a vertex r in G, and for each $i \geq 0$, define L_i as the set of vertices of G at distance exactly i from r (the distance between two vertices is the minimum number of edges on a path connecting the two vertices). Note that the sets L_i partition the vertex set of G. We now define a 2-coloring of G as follows: all the vertices of the sets L_i with i odd are assigned color 1, and all the vertices of the sets L_i with i even are assigned color 2.

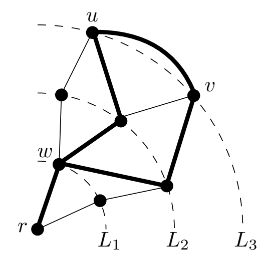

Figure 1: An illustration of the proof of Theorem 2.

We now prove that this is indeed a 2-coloring (assuming that G has no odd cycles). To this end, consider an edge uv of G (and assume by symmetry that the distance between r and u is at most the distance between r and v), and observe that by the definition of $(L_i)_{i\geq 0}$, either u and v both lie in some set L_i, or $u \in L_i$ and $v \in L_{i+1}$. In the second case, it follows from the definition of our coloring that u and v receive different colors. In the first case, consider a shortest path P_u between u and r, and a shortest path P_v between v and r. Let w be the vertex of $P_u \cap P_v$ that is the furthest from r (note that possibly $z = w$ if $P_u \cap P_v$ only consists of $\{r\}$). Now observe that the edge uv, together with the subpath of P_v between v and w, and the the subpath of P_u between u and w, forms an odd cycle (see Figure 1 for an illustration), which is a contradiction. □

It can be checked that the proof actually gives a polynomial algorithm to decide whether a graph is bipartite (and find a 2-coloring if this is the case, or an odd cycle otherwise). On the other hand, deciding whether the chromatic number of a graph is at most 3 is an NP-complete problem (even in very simple classes of graphs).

2 Mycielski's construction

We define a sequence $(M_k)_{k\geq 1}$ of graphs inductively. M_1 is a single vertex, and M_2 consists of two vertices joined by an edge. For $k \geq 3$, M_k is constructed as follows: we start with a copy of M_{k-1}, and for each vertex v in this copy

of M_{k-1}, we add a vertex v' that has precisely the same neighbors as v (we say that v' is the twin of v). Finally, we add a vertex z^* that is adjacent to all the newly created vertices v', and non-adjacent to all the vertices of the copy of M_{k-1}.
It is not difficult to check that M_3 is a 5-cycle and M_4 is the so-called Mycielski graph, depicted below.

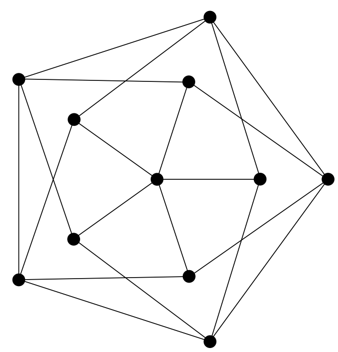

Figure 2: The graph M_4.

We now prove the following theorem.

Theorem 3. *For any $k \geq 1$, M_k is triangle-free (i.e. $\omega(G) \leq 2$) and $\chi(G) = k$.*

Proof. We prove the theorem by induction on k.
We start by proving that M_k has no triangle. This is clear if $k \leq 2$, so assume that $k \geq 3$. Let us denote by S the set of newly created vertices distinct from z^*. Assume for the sake of contradiction that there exist a triangle T in M_k. Since S is a stable set and M_{k-1} is triangle-free (by induction), T has two vertices in the copy of M_{k-1} (call them u, v) and one in S (call it w'). But since w' has the same neighbors in the copy of M_{k-1} as its twin w, uvw forms a triangle in M_{k-1}, which contradicts the induction hypothesis.

We now prove that for any $k \geq 3$, $\chi(M_k) = k$. The cases $k = 1$ and $k = 2$ are clear, so we can assume that $k \geq 3$. Since the copy of M_{k-1} is $(k-1)$-colorable (by induction), we can color it with colors $1, 2, \ldots, k-1$, then use color k for the vertices of S, and finally color 1 for z^*. This shows that $\chi(M_k) \leq k$. It remains to prove that $\chi(M_k) \geq k$. For this we will need the following simple observation.

For any graph H and any coloring of H with $\chi(H)$ colors, each color class contains a vertex that is adjacent to all the other color classes. (1)

To see why this holds, just observe that the negation of (1) implies that there is a color, say i, such that each vertex colored i is not adjacent to some other color class. In this case it is possible to recolor each vertex colored i with another color. But this results in a coloring of H with at most $\chi(H) - 1$ colors, which is impossible.

Now, assume for the sake of contradiction that M_k has a coloring with $k-1$ colors. Since $\chi(M_{k-1}) = k-1$ (by induction), we can apply (1) to the copy of M_{k-1} in M_k. This gives us sequence of vertices $v_1, v_2, \ldots, v_{k-1}$ in the copy of M_{k-1}, such that each v_i is colored i and is adjacent to all the other color classes. In particular this implies that for each i, the twin v_i' of v_i is also colored i. But then the vertex z^* is adjacent to vertices of colors $1, 2, \ldots, k-1$, which is a contradiction. □

3 Chromatic number and maximum degree

The *degree* of a vertex v (usually denoted by $d(v)$) in a graph G is the number of neighbors of v in G.

We now consider the following *greedy algorithm* to obtain a coloring of a graph G.

Order the vertices as $v_1, v_2, \ldots, v_n$. For $i = 1$ to n, color v_i with the smallest color (recall that colors are positive integers) that does not yet appear on its neighborhood.

Note that when choosing the color of a vertex v, at most $d(v)$ colors are forbidden to v and in particular, if v has at least $d(v) + 1$ available choices then it can always find a color that does not appear in its neighborhood. This shows the following.

Observation 4. *For any graph G with maximum degree Δ, the greedy algorithm finds a coloring with at most $\Delta + 1$ colors, and in particular $\chi(G) \leq \Delta + 1$.*

Note that this is best possible: odd cycles and complete graphs satisfy the bound above with equality. However, the following result (that we won't prove during the lecture) shows that these are the only extremal examples.

Theorem 5 (Brooks Theorem)**.** *If G is a connected graph with maximum degree at most Δ, distinct from an odd cycle or a complete graph, then $\chi(G) \leq \Delta$.*

We conclude with an exercise showing that the greedy algorithm can perform quite poorly for some instances.

Exercise 1. Consider the graph G_n with vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$, in which each vertex u_i is adjacent to all the vertices v_j with $j \neq i$ (see Figure 3 for a picture of G_4).

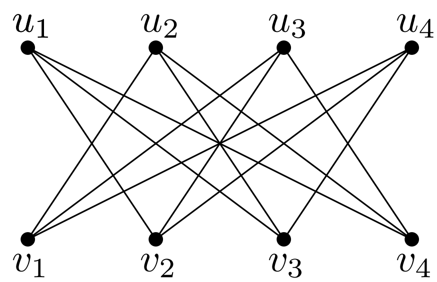

Figure 3: The graph G_4.

Show that $\chi(G_n) = 2$ for any $n \geq 2$. Show that there is a an order on the vertices of G_n such that the greedy coloring in this order uses n colors.