ORCO – Graphs and Discrete Structures
October 6, 2021 – Lecture 2

1 Degeneracy

We say that a graph G is *k-degenerate* (for some integer $k \geq 0$) if G has an ordering $v_1, \ldots, v_n$ of its vertices, such that for any i, the number of neighbors v_j of v_i with $j < i$ is at most k.
The same proof as is the first lecture (for bounded degree graphs), shows that the greedy algorithm performs very well on k-degenerate graphs (using the same vertex ordering).

Observation 1. *If G is k-degenerate, then $\chi(G) \leq k+1$.*

Homework – Find graphs for which equality holds, other than complete graphs and odd cycles.

In many applications we will need a slightly different (but equivalent) definition of k-degeneracy.
Before that, let us recall the notions of subgraphs and induced subgraphs. Given a graph $G = (V, E)$, we say that a graph $H = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. Moreover, we say that H is an *induced subgraph* of G if E' consists of all the edges of E with both endpoints in V'. You can think of a subgraph of G as a graph obtained from G by removing any set of vertices and edges. On the other hand, an induced subgraph of G is a graph obtained from G by only removing vertices (and the edges containing these vertices).

Theorem 2. *A graph G is k-degenerate if and only if each induced subgraph H of G contains a vertex of degree at most k in H.*

Proof. Assume first that G is k-degenerate. By definition, G has an ordering $v_1, \ldots, v_n$ of its vertices such that each vertex v_i has at most k neighbors v_j with $j < i$. Consider any induced subgraph H of G, and let S be the subset of $\{1, \ldots, n\}$ such that the vertices of H are precisely the vertices v_i with $i \in S$. Let ℓ be the maximum of S. Then v_ℓ has at most k neighbors v_j with $j < k$, and in particular v_ℓ has at most k neighbors in the set $\{v_i \,|\, i \in S\}$. It follows that v_ℓ has degree at most k in H, as desired.
Assume now that each induced subgraph H of G contains a vertex of degree at most k in H. In particular, G itself has a vertex of degree at most k

(call it v_n). For $i = n-1, \ldots, 1$, we define v_i as a vertex of degree at most k in $G \setminus \{v_n, \ldots, v_{i+1}\}$ (such a vertex exists since any induced subgraph of G has a vertex of degree at most k). In this ordering, observe that for any i, the number of neighbors v_j of v_i with $j < i$ is precisely the degree of v_i in $G \setminus \{v_n, \ldots, v_{i+1}\}$, which is at most k by definition. It follows that G is k-degenerate, which concludes the proof. $\square$

A class of graphs $\mathcal{C}$ is *hereditary* if it is closed under taking induced subgraphs (i.e. any induced subgraph of a graph of $\mathcal{C}$ is also in $\mathcal{C}$). The following simple corollary of Observation 1 and Theorem 2 has many applications.

Corollary 3. *Assume that $\mathcal{C}$ is a hereditary class such that any graph of $\mathcal{C}$ has a vertex of degree at most k. Then for any graph G of $\mathcal{C}$, $\chi(G) \leq k+1$.*

Proof. Let G be a graph of $\mathcal{C}$. Since any induced subgraph of G is in $\mathcal{C}$, any induced subgraph of G has a vertex of degree at mot k, and thus by Theorem 2, G is k-degenerate. It then follows from Observation 1 that $\chi(G) \leq k+1$, as desired. $\square$

2 Planar graphs

A graph is *planar* if it has an embedding in the plane with no edge-crossings. The connected component of the plane minus the embedding are called the *faces*. A fundamental result about planar graphs is the following.

Theorem 4 (Euler's Formula)**.** *If G is a connected planar graph, embedded in the plane, with n vertices, m edges, and f faces, then $n - m + f = 2$.*

Note that it shows in particular that the number of faces of a planar graph does not depend on the embedding the graph (and thus we can remove "embedded in the plane" in the theorem above).

We will not prove Euler's formula (during the lectures I showed you why it holds when the graph has a straight-line drawing such that each face is a convex polygon, but the purpose was to give you an intuition of why this holds, not to prove it formally).

We will now deduce the following simple result from Euler's Formula.

Lemma 5. *Any planar graph on $n \geq 3$ vertices has at most $3n-6$ edges.*

Proof. We can assume that the graph is connected (otherwise we consider each connected component separately). Let m be the number of edges and f be the number of faces of G. By Euler's Formula, we have $n - m + f = 2$ and thus $f = 2 - n + m$. A simple counting argument shows that the sum of the degrees (number of edges in a boundary walk) of the faces of G is equal to $2m$, and since each face has degree at least 3, we have $2m \geq 3f$ and thus $f \leq \frac{2}{3}m$. It follows that $2 - n + m \leq \frac{2}{3}m$ and thus $m \leq 3n - 6$, as desired. $\square$

Homework – Similarly, find a bound on the number of edges of a triangle-free planar graph.

Recall that the sum of the degrees of the vertices of a graph is precisely twice the number of edges of that graph. Thus, it follows from Lemma 5 that any planar graph has average degree less than 6. In particular:

Corollary 6. *Any planar graph has a vertex of degree at most 5.*

Homework – Similarly, find a bound on the minimum degree of a triangle-free planar graph.

Since any induced subgraph of a planar graph is planar, the class of planar graphs is hereditary, and thus a direct consequence of Corollaries 6 and 3 is the following simple result.

Corollary 7. *Any planar graph has chromatic number at most 6.*

Perhaps the most important result in graph coloring is the following improved version, solving a problem posed by Guthrie in 1852.

Theorem 8 (The 4 Color Theorem)**.** *Any planar graph has chromatic number at most 4.*

We will not prove it here, but next week we will prove the following version instead (that goes halfways between Corollary 7 and Theorem 8.

Theorem 9 (The 5 Color Theorem)**.** *Any planar graph G has chromatic number at most 5.*