ORCO – Graphs and Discrete Structures
October 13, 2021 – Lecture 3

1 Coloring planar graphs

Recall that a graph is *planar* if it has an embedding in the plane with no edge-crossings.
Last time we proved that any planar graph has a vertex of degree at most 5, and as a consequence, planar graphs are 6-colorable.
We now prove the following improved version.

Theorem 1 (The 5 Color Theorem)**.** *Any planar graph G has chromatic number at most 5.*

Proof. We prove the theorem by induction on the number of vertices. The conclusion is clear if G has at most 1 vertex, so assume G has at least 2 vertices. Consider any fixed embedding of G in the plane (without edge-crossings). The first case is that G contains a vertex v of degree at most 4. Then we can color $G - v$ by induction with at most 5 colors, and since v has degree at most 4, we can extend the coloring to v (v has 5 choices and at most 4 are forbidden). Since every planar graph contains a vertex of degree at most 5, the last case is that G contains a vertex v of degree precisely 5. Again, we color $G - v$ by induction and attempt to extend this coloring to v. If some color (among $1, \ldots, 5$) does not appear among the neighbors of v, we can extend the coloring to v. Otherwise, we can assume by symmetry that the neighbors $v_1, \ldots, v_5$ (in clockwise order around v) are colored such that v_i has color i for any $1 \leq i \leq 5$. Let G_{13} be the set of vertices of G colored 1 or 3, and let C_{13} be the connected component of G_{13} containing v_1. In C_{13}, interchange colors 1 and 3 (i.e. recolor all vertices colored 1 with color 3, and all vertices colored 3 with color 1). The resulting coloring is still a 5-coloring of $G - v$, and if $v_3 \notin C_{13}$, color 1 does not appear in the neighborhood of v and we can extend the coloring to v. Assume now that $v_3 \in C_{13}$ (which implies that there is a path P_{13} of vertices colored 1 or 3 between v_1 and v_3 in $G - v$). Now do exactly the same thing with v_2 and v_4. Again, we can remove color 2 from the neighborhood of v unless there is a path P_{24} colored 2 or 4 between v_2 and v_4 in $G - v$. But since $v_1, \ldots, v_5$ are in clockwise order around v, the paths P_{13} and P_{24} must intersect. As they are vertex disjoint, two edges of P_{13} and P_{24} must cross, which contradicts the fact that the embedding of G was planar. $\square$

2 Planar graph drawing

In the remainder of the lecture we will prove Fáry's theorem (originally due to Wagner, 1936), which states that for any planar drawing of a planar graph there is an equivalent planar drawing in which all edges are straight-line segments.

We will first need a couple of ingredients. The first is a simple observation that can be deduced from the fact that any n-vertex planar has at most $3n - 6$ edges (assuming $n \geq 3$), proved in the last lecture.

Lemma 2. *Every planar graph G on $n \geq 4$ vertices has at least 4 vertices of degree at most 5.*

Proof. Note that if some graph obtained from G by adding edges has at least 4 vertices of degree at most 5, then G also has at least 4 vertices of degree at most 5. Hence, we can assume without loss of generality that G is edge-maximal with respect to being planar, and in particular G has minimum degree at least 3. Assume for the sake of contradiction that G has at most 3 vertices of degree at most 5. So $n - 3$ vertices have degree at least 6, and the remaining 3 have degree at least 3. It follows that the sum of degrees in G is at least $6(n - 3) + 3 \cdot 3 = 6n - 9$, and thus G has at least $3n - 4$ edges, a contradiction. $\square$

A planar graph is said to be *outerplanar* if it has a planar drawing in which all vertices lie on the outerface.

Lemma 3. *Every outerplanar graph G contains a vertex of degree 2.*

Proof. We can again assume that G is edge-maximal with respect to being outerplanar graph. In particular, G has no cut-vertices, and if we consider any planar drawing of G in which all vertices lie on the outerface, the outerface is a cycle (without repeated vertices), and all internal faces are triangles. The *weak dual* of (some planar drawing of) G is the graph whose vertices are the internal faces of G, and such that two vertices are adjacent if and only if the corresponding faces of G share an edge. It is not difficult to check that the weak dual of an edge-maximal outerplanar graph is a tree (if it contains a cycle, then some vertex of G does not lie on the outerface). Note that every tree has a vertex of degree 1, and the corresponding triangular face of G must share two edges with the outerface. It follows that this face contains a vertex of degree 1, as desired. $\square$

Since outerplanar graphs form a hereditary family, this shows that outerplanar graphs are 2-degenerate. We have the following immediate corollary.

Corollary 4. *Every outerplanar graph is 3-colorable.*

In the *Art gallery problem* we are given an n-vertex polygon in the plane (an n-gon, in short), and the goal is to place guards inside the n-gon so that all the (inside) region bounded by the n-gon is under the surveillance of some guard (a point is under the surveillance of a guard if the segment connecting them is contained in the region bounded by the n-gon).

Theorem 5. $\lfloor n/3 \rfloor$ *guards are sufficient to guard an n-gon.*

Proof. Arbitrarily add edges to the internal faces of the n-gon until all internal faces are triangles. We obtain an n-vertex outerplanar graph G, which is 3-colorable by Corollary 4, and thus some color $i \in \{1, 2, 3\}$ is such that at most $\lfloor n/3 \rfloor$ vertices of G are colored i. Note that all triangles of G are colored $1, 2, 3$, and thus if we place the guards at the location of the vertices colored i, all the region bounded by the n-gon is guarded. $\square$

We will only need the case $n \geq 5$ of Theorem 5, which can be rephrased as follows.

Corollary 6. *Any n-gon with $3 \leq n \leq 5$ has an interior point x, such that each segment connecting x to the vertices of the n-gon is contained in the region bounded by the n-gon.*

We are now ready to prove Fáry's theorem.

Theorem 7. *For every planar embedding of a planar graph G, there is an equivalent embedding in which all edges are straight-line segments.*

Proof. As before, we can assume without loss of generality that G is edge-maximal with respect to the property of being planar (if we can draw a supergraph of G with edges as straight-line segments, then G certainly also has such a drawing). In particular, the outerface is a triangle, and for every vertex v, there is a circular order on the neighbors of v such that any two consecutive neighbors in this order are adjacent. It follows that if we remove v, then these neighbors form a face in the corresponding embedding of $G\backslash\{v\}$. Let us prove the result by induction on the number n of vertices. This is certainly true if $n \geq 3$ (in this we only have a single vertex, a single edge, or a triangle), so assume that $n \geq 4$. Since the outerface is a triangle, if follows from Lemma 2 that G contains a vertex v of degree at most 5 that does not lie on the outerface of G. We remove v and apply induction on the resulting embedding of $G \setminus \{v\}$. We obtain an equivalent a straight-line drawing of $G \setminus \{v\}$ in which the neighbors of v form a face (and thus a k-gon, for $3 \leq k \leq 5$). It follows from Corollary 6 that we can add v inside the region bounded by the k-gon, and connect v to its neighbors with straight-line segments without creating crossings. $\square$