**ORCO – Graphs and Discrete Structures**
**October 20, 2021 – Lecture 4**

# 1 Interval graphs

Recall that for any graph $G$, $\omega(G) \leq \chi(G)$. In this lecture we are going to see natural graph classes for which equality holds.

A graph $G$ with vertices $v_1, \ldots, v_n$ is said to be an *interval graph* if there exist intervals $I_1, \ldots, I_n$, such that any two vertices $v_i$ and $v_j$ are adjacent in $G$ if and only if the corresponding intervals $I_i$ and $I_j$ intersect. The interval $I_1, \ldots, I_n$ are called *the interval representation* of $G$.

In such a graph $G$, consider the interval $I_i$ with rightmost left end. Note that by definition, all the neighbors of the corresponding vertex $v_i$ also intersect this left end, and thus $v_i$, together with its neighbors, forms a clique.

**Observation 1.** *Any interval graph $G$ contains a vertex $v$ whose neighborhood is a clique. In particular, $v$ has degree at most $\omega(G) - 1$.*

Note that each induced subgraph of an interval graph is also an interval graph (starting with an interval representation of the graph and removing some intervals, we can get interval representations of any induced subgraph of $G$). Applying Corollary 3 of the Lecture notes of the 2nd lecture, we obtain:

**Theorem 2.** *Any interval graph $G$ has chromatic number $\omega(G)$.*

Another (equivalent) way to look at this theorem (perhaps more algorithmic) is the following.
Given an interval graph $G$ with interval representation $I_1, \ldots, I_n$, sort the interval by their left end and color them using the greedy algorithm from left to right. Then the observations above shows that the resulting coloring is a coloring with $\omega(G)$ colors.

# 2 Chordal graphs

An equivalent way to define interval graphs would be to say that the interval $I_1, \ldots, I_n$ are subpaths of a given path. This yields the following natural generalisation.

A graph $G$ with vertices $v_1, \ldots, v_n$ is a *chordal graph* if there is a tree $\mathcal{T}$ and subtrees $T_1, \ldots, T_n$ of $\mathcal{T}$ such that for any $i, j$, $v_i$ and $v_j$ are adjacent in $G$ if and only if $T_i$ and $T_j$ intersect.

While this definition is quite natural, it is usually easier to work with the following equivalent definition.

Given a graph $G$, a *subtree representation* of $G$ is a tree $\mathcal{T}$ together with sets $(B_t)_{t\in\mathcal{T}}$ of vertices of $G$ (the set $B_t$ is called the *bag* of $t$), with the following properties.

1. $u$ and $v$ are adjacent in $G$ if and only if there is a bag $B_t$ containing $u$ and $v$ in the subtree representation, and
2. for any vertex $v$, the set of nodes $t$ of $\mathcal{T}$ whose bag $B_t$ contains $v$ forms a subtree of $\mathcal{T}$ (equivalently, if $v$ is in two bags $B_t$ and $B_s$, then $v$ lies in all the bags $B_{t'}$ such that $t'$ is on the path between $t$ and $s$ in $\mathcal{T}$).

Now it can be seen that a graph is a chordal graph if and only if it has a subtree representation. To see why this holds, if you start with some subtrees $T_1, \ldots, T_n$ of a tree $\mathcal{T}$ as above, then for any node $t$ of $\mathcal{T}$ you can define $B_t$ as the set of vertices $v_i$ such that the corresponding subtree $T_i$ contains $t$ (it is not difficult to check that the definition of a subtree representation is then satisfied). On the other hand, starting with a subtree representation $\mathcal{T}$ with bags $(B_t)_{t\in\mathcal{T}}$, we can define the subtree $T_i$ of a vertex $v_i$ as the set of nodes $t$ of $\mathcal{T}$ such that $B_t$ contains $v_i$. In this case condition 1 in the definition above implies that two vertices are adjacent if and only if the corresponding subtrees intersect.

It follows directly from the definition that any interval graph is a chordal graph (and in fact, the interval graphs are precisely the chordal graphs with a subtree representation that is a path), but a natural question is whether there are chordal graphs that are not interval graphs (that is, we have really defined a more general class of graphs).
We start by proving the following.

**Proposition 3.** *Trees are chordal graphs.*

*Proof.* We prove the result by induction of the number of vertices of a tree $T$. The result is clear if $T$ contains a single vertex, so assume it contains at least two vertices. Let $u$ be a leaf of $T$, and let $v$ be its unique neighbor in $T$.

By induction, find a subtree representation of $T \setminus \{u\}$, say with underlying tree $\mathcal{T}$ and bags $(B_t)_{t\in\mathcal{T}}$. Consider a node $t \in \mathcal{T}$ such that $v \in B_t$, and add a new leaf $t^*$ in $\mathcal{T}$ whose unique neighbor is $t$. Define $B_{t^*} = \{u, v\}$. We now have a subtree representation of $T$, as desired. $\square$

It can be checked that the graph of Figure 1 is not an interval graph. However it is a tree, and it is thus chordal by Proposition 3.

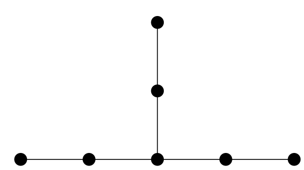

Figure 1: A chordal graph which is not an interval graph.

Given a chordal graph $G$, let us say that a subtree representation $\mathcal{T}, (B_t)_{t\in\mathcal{T}}$ of $G$ is *pruned* if for any leaf $t$ of $\mathcal{T}$, the bag $B_t$ of $t$ contains a vertex of $G$ that does not appear in any other bag.

**Proposition 4.** *Any chordal graph has a pruned subtree representation.*

*Proof.* Assume some leaf $t$ is such that all the vertices of $B_t$ appear in another bag. Then by connectivity, all the vertices of $B_t$ also appear in the bag $B_s$ of the unique neighbor $s$ of $t$ in $\mathcal{T}$. In this case we can simply delete $t$ from $\mathcal{T}$ and the graph it describes remains unchanged. If we repeat this procedure until no such leaf $t$ exists we have the property that each leaf contains a vertex that does not appear in another bag, as desired. $\square$

Pruned subtree representations are very useful. Let us now prove the following generalisation of Observation 1.

**Proposition 5.** *Any chordal graph $G$ contains a vertex $v$ whose neighborhood is a clique. In particular, $v$ has degree at most $\omega(G) - 1$.*

*Proof.* Consider a pruned subtree representation $\mathcal{T}$ of $G$, with bags $(B_t)_{t\in\mathcal{T}}$, and a leaf $t$ of $\mathcal{T}$. By the definition of a pruned subtree representation, $B_t$ contains a vertex $v$ that does not appear in another bag. Since for any neighbor $u$ of $v$, $u$ and $v$ have to be in a common bag, it follows that $B_t$ contains all the neighbors of $v$, and thus $v$ and its neighbors forms a clique. $\square$

As in the case of interval graphs, observe that each induced subgraph of a chordal graph is also a chordal graph (starting with a subtree representation of the graph and removing some subtrees, we can get subtree representations of any induced subgraph of $G$). Applying Corollary 4 of the Lecture notes of the 2nd lecture, we obtain:

**Theorem 6.** *Any chordal graph $G$ has chromatic number $\omega(G)$.*

# 3 Perfect graph

A graph $G$ is perfect if and only if any induced subgraph $H$ of $G$ satisfies $\chi(H) = \omega(H)$.

What we have proved in the previous section is that chordal graphs (and interval graphs) are perfect.

Example of *non* perfect graphs include odd cycles, and their complements.

*Homework – prove that complements of odd cycles are not perfect.*

# 4 Homework

The goal of this problem is to prove that a graph is chordal if and only if it does not contain an induced cycle of length at least 4 (in other words, every cycle of length at least 4 has a chord).

We say that a vertex $v$ is *simplicial* if the neighborhood of $v$ is a clique. We have proved above that any chordal graph has a simplicial vertex. Since each induced subgraph of a chordal graph is a chordal graph, we have the stronger property that if $G$ is a chordal graph, any induced subgraph of $G$ has a simplicial vertex.
It turns out to be an equivalence (but we won't prove it, you can just use the following theorem freely).

**Theorem 7.** *A graph is chordal $G$ if and only if any induced subgraph of $G$ has a simplicial vertex.*

(1) Prove that if $G$ is chordal, then $G$ does not contain an induced cycle of length at least 4.

(2) We now want to prove the converse, that is: *if $G$ has no induced cycle of length at least 4, then $G$ is chordal.* Given two non-adjacent vertices $x, y$ of a graph $G$, an $(x, y)$-*separator* is a set $S$ of vertices such that $x$ and $y$ are in distinct connected components of $G - S$. A $(x, y)$-separator $S$ is *minimal* if no subset of $S$ is an $(x, y)$-separator.
Prove that such a minimal $(x, y)$-separator always exists.

(3) Prove that if $S$ is a minimal $(x, y)$-separator, every vertex of $S$ has a neighbor in the connected component of $G-S$ containing $x$ (and similarly for $y$).

(4) Prove that if $G$ has no induced cycle of length at least 4, and $x$, $y$ are non-adjacent vertices in the same connected component of $G$, then any minimal $(x, y)$-separator is a clique.
*Hint: assume by contradiction that the separator $S$ has a non-edge $u, v$ and use it to construct an induced cycle passing through $u, v$ that is partly in the component of $G - S$ containing $x$, and partly in the component of $G - S$ containing $y$.*

(5) Use the previous question to prove the following by induction: *if $G$ has no induced cycle of length at least 4, then $G$ is either a complete graph, or it contains two non-adjacent simplicial vertices.* Conclude.