ORCO – Graphs and Discrete Structures
November 10, 2021 – Lecture 6

In this lecture, we start to investigate extremal questions for graphs. For a positive integer n and a graph G, let us denote by $ex(n, G)$ the largest number of edges in an n-vertex graph H that does not contain G as a subgraph.

1 Excluding a clique

The following result of Turán is a precursor of the field of extremal graph theory (and extremal combinatorics in general). It studies $ex(n, K_r)$. For positive integer r and positive integers $s_1, \ldots, s_r$ let $K_{s_1,\ldots,s_r}$ stand for the *complete r-partite graph* with part sizes $s_1, \ldots, s_r$, i.e., a graph with parts of sizes $s_1, \ldots, s_r$, no edges inside one part and all possible edges drawn between different parts. (If not necessarily all possible edges are drawn between different parts then we call such a graph *r-partite*.) We say that an r-partite graph *has almost equal parts* if sizes of any two parts differ by at most 1.

Theorem 1 (Turán)**.** *If a graph G on n vertices has no K_{r+1} as a subgraph, then G has at most as many edges as an n-vertex complete r-partite graph with almost equal parts. (Moreover, the equality is only possible for complete r-partite graphs with almost equal parts.) In particular, $ex(n, K_r) \leq (1-\frac{1}{r})\frac{n^2}{2}$.*

Proof. This proof is via Zykov symmetrization. The idea is to gradually change the structure of the graph while preserving its properties and increasing the number of edges.
Let G be the graph as in the theorem with $ex(n, K_r)$ edges. First, note that if vertices v and w are non-adjacent then $d(v) = d(w)$. Indeed, if $d(v) > d(w)$ then we can remove the vertex w and replace it by a 'copy' of v, i.e., a vertex v' that has the same adjacencies and non-adjacencies as v. Note that no copy of K_{r+1} may appear after such 'copying' because any clique can either contain v or v', and thus if there is a clique with v' then there is a clique of the same size with v. Also note that the number of edges in the new graph is strictly larger.
Second, we show that non-adjacency $\not\sim$ is an equivalence relation: if $v \not\sim w$ and $u \not\sim w$ then $v \not\sim u$. Note that $d(v) = d(w) = d(u)$ by the previous paragraph. However, if v and u are adjacent, then the edge vu is counted in the degrees of both v and u. Thus, if we remove both v and u and replace

them by two 'copies' of w then we increase the number of edges in the graph by 1 without creating a copy of K_{r+1}.

Since $\not\sim$ is an equivalence relation, the vertex set of G can be split in parts $V_1, \dots, V_k$, each of which is an independent set and all edges are drawn between any two parts. Indeed, if there is at least 1 edge missing between V_i and V_j then the vertices of $V_i \cup V_j$ belong to the same class. We conclude that G is a complete multipartite graph with k parts. Note that $k \leq r$, otherwise we have a copy of K_{r+1}.

The last part of the proof is to show that complete r-partite graphs with almost equal parts maximize the number of edges. First, if the number of parts k satisfy $k < r$, then we can split the largest part into two and draw all edges between the parts, increasing the number of edges. Thus, $k = r$. Next, assume that, say $|A_i| \geq |A_j| + 2$. Then move one vertex from A_i to A_j, forming another complete r-partite graph. Note that, when moving, we deleted $|A_j|$ edges and added $|A_i| - 1 \geq |A_j| + 1$ edges. Thus, we have increased the total number of edges.

It is also easy to see that this proof also implies that all extremal graphs are complete r-partite graphs with almost equal parts. □

Once the extremal problem is solved, an important class of questions to ask is how stable is the extremum. In this context, we can vaguely formulate it as follows: is it true that if the number of edges in an n-vertex K_{r+1}-free graph is close to the extremum, then the graph itself is close to an r-partite graph?

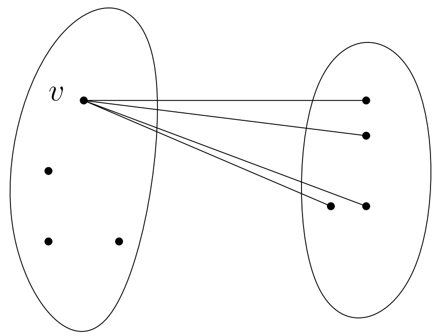
In this case, the answer is given by the following result (which we present only for $r = 2$ for the sake of simplicity).

Theorem 2. *Let $G = (V, E)$ be a graph on n vertices, such that G does not contain K_3 as subgraph and $|E| = ex(n, K_3) - t$ for some non-negative integer t. Then one can remove at most t edges from G to make it bipartite.*

Proof. Let $v \in V$ be a vertex of maximum degree in G and let us denote its degree by Δ.

We denote by $N(v)$ the set of neighbors of v in G, and we define $B := N(v)$ and $A := V \backslash N(v)$. Since v is of maximum degree, we have that $|B| = \Delta$ and $|A| = n - \Delta$.

For any subgraphs H_1 and H_2 of G, we denote by $E(H_1, H_2)$ the set of edges of G with one extremity in H_1 and the other in H_2. We use the notation $E(H)$ for $E(H, H)$.



$A := V \backslash N(V)$ $\qquad$ $B := N(V)$

Let us first observe that there are no edges with both endpoints in B. Indeed, let us assume that there are two vertices $v_1, v_2 \in B$ such that (v_1, v_2) is an edge in G. Then the vertices v, v_1, v_2 form a K_3.

We prove now that $|E(A)| \leq t$. If this is indeed the case, the deletion of the edges in $E(A)$ will provide us a bipartite graph with bipartition $A \cup B$ on the same set of vertices as G (remember that there are no edges running between vertices of B, and after deletion there will be no edges running between vertices of A).

Since the cardinality of A is $n - \Delta$, and the degree of each vertex $u \in A$ is at most Δ, we have that

$$\Delta(n - \Delta) \geq \sum_{u \in A} d(u).$$

On the other hand, we have that

$$\sum_{u \in A} d(u) = 2|E(A)| + |E(A, B)| =$$

$$= |E(A)| + (|E(A) + |E(A, B)|) = |E(A)| + |E(G)|.$$

Since $|E(G)| = ex(n, 3) - t$, and the number of edges in $K_{\Delta, n-\Delta}$ (which is $\Delta(n - \Delta)$) cannot exceed $ex(n, K_3)$, we have that $|E(G)| \geq \Delta(n - \Delta) - t$. This, together with the two inequalities above, imply that $|E(A)| \leq t$. We can now remove all the edges of $E(A)$ to obtain a bipartite graph, which completes the proof. $\square$