ORCO – Graphs and Discrete Structures
November 17, 2021 – Lecture 7

A natural question is whether a result similar to Turán's theorem holds if we exclude an arbitrary subgraph H instead of K_{r+1}. The next theorem (which we will not prove here) shows that the answer only depends on the chromatic number of H.

Theorem 1 (Erdős-Stone)**.** *For any graph H we have $ex(n,H) \leq (1 - \frac{1}{\chi(H)-1})\frac{n^2}{2} + o(n^2)$.*

Since $\chi(K_{r+1}) = r+1$, this immediately implies the previous theorem up to lower order terms.
Observe that when $\chi(H) = 2$ (i.e. when H is bipartite), this theorem implies that any graph on n vertices excluding H as a subgraph has $o(n^2)$ edges. In the next section, we consider the case where H is a 4-cycle (which is clearly bipartite), and we find the right order of magnitude for the number of edges in graphs excluding H.

1 Excluding a 4-cycle

Theorem 2. *We have $ex(n, C_4) \leq \frac{1}{4}\sqrt{4n^3 - 3n^2} = (1 + o(1))\frac{n^{3/2}}{2}$.*

Proof. Let m be the number of edges of an n-vertex graph G that has no 4-cycle. A *cherry* in G is a pair $(v, \{u, v\})$ where u, v, w are vertices of G and $\{u, v\}$ is an unordered pair of neighbors of v in G. If we fix v there are $\binom{d(v)}{2}$ pairs $\{u, w\}$ such that $(v, \{u, v\})$ is a cherry, so $K = \sum_v \binom{d(v)}{2}$. On the other hand, if we fix $\{u, v\}$ there is at most one vertex v such that $(v, \{u, v\})$ is a cherry (if there was a second vertex with this property, then the graph would contain a 4-cycle). This shows that $\sum_v \binom{d(v)}{2} = K \leq \binom{n}{2}$. Thus, we have

$$\sum_v d(v)^2 - 2m = \sum_v (d(v) - 1)d(v) \leq n(n-1).$$

By convexity of the function $x \mapsto x^2$ (or by applying Cauchy-Schwarz inequality), it follows that $\sum_v d(v)^2 \geq \frac{1}{n}(\sum_v d(v))^2 = \frac{1}{n}(2m)^2$. As a consequence, the displayed inequality implies the following quadratic inequality: $(2m)^2 - 2mn \leq n^2(n-1)$. Solving it with respect to the variable $2m$, we get $2m \leq \frac{1}{2}\sqrt{4n^3 - 3n^2}$. $\square$

We now construct examples showing that the bound given in the previous theorem is close from best possible.
A *projective plane of order* q is a set of elements (called *points*), together with a collection of sets of points (called *lines*) with the following properties.

1. there are q^2+q+1 points and q^2+q+1 lines
2. each point is in $q+1$ lines and each line contains $q+1$ points
3. for any pair of point there is a unique line containing them, and any pair of lines intersect in a unique point.

It is known that projective planes of order q exist whenever q is a prime power. Proving that they exist (or don't exist) for other values of q is an important open problem.
Given a projective plane, we can construct a bipartite graph as follows: the vertices are the points and lines of the projective plane, and for any point p and line ℓ such that $p \in \ell$, we add an edge between p and ℓ in the graph.
Note that the resulting bipartite graph has no 4-cycle (otherwise two points would be contained in two lines, contradicting property 3 above). Moreover, if we start with a projective plane of order q, the bipartite graph has $n = 2q^2+2q+2 \approx 2q^2$ vertices and $m = (q+1)(q^2+q+1) \approx q^3$ edges (since each vertex has degree $q+1$, by property 2 above). It follows that $m \approx (n/2)^{3/2}$ which is very close from the bound of Theorem 2.

2 graphs with no short cycles and large chromatic number

In this section we will see a way to prove the existence of graphs with no triangles or 4-cycles, but with large chromatic number. The downside is that the proof is only existential (we don't construct such graphs explicitly). Such graphs are, however, very difficult to construct explicitly. Moreover, the proof easily extends to the following: *for any g and k, there exists a graph of chromatic number at least k that has no cycles of length less than g.* The homework assignment for next week is to see how to modify the proof below to obtain such a result.

We will need the following simple probabilistic results.

Lemma 3 (Markov Inequality)**.** *If $X \geq 0$ is a discrete random variable, then for any $t > 0$, $\mathbb{P}(X \geq t) \leq \mathbb{E}(X)/t$.*

Proof. By definition,

$$\mathbb{E}(X) = \sum_{x} x\,\mathbb{P}(X = x) \geq \sum_{x \geq t} x\,\mathbb{P}(X = x) \geq t \sum_{x \geq t} \mathbb{P}(X = x) = t\,\mathbb{P}(X \geq t),$$

where the first inequality follows from the fact that $X \geq 0$. It follows that $\mathbb{P}(X \geq t) \leq \mathbb{E}(X)/t$, as desired. □

For integer random variables, Markov inequality has the following simple (yet very useful) consequence (just take $t = 1$).

Corollary 4. *If $X \geq 0$ is an integer random variable, then $\mathbb{P}(X \neq 0) \leq \mathbb{E}(X)$.*

To construct our triangle-free graphs of large chromatic number, we will need the notion of a *random graph*. For an integer n and a real $p \in [0, 1]$ (that might depend on n), let $G(n, p)$ be the (random) graph with n vertices, in which we add an edge with probability p, independently, between any pair of vertices.

Homework

1. For an arbitrary graph H calculate the expected number of copies of H in $G(n, p)$. The expression should depend on the number of automorphisms of H.

2. Prove that the random graph $G(n, \frac{1}{2})$ contains no clique of size $2 \log n$ and no stable set of size $2 \log n$, with probability tending to 1 as $n \to \infty$.