ORCO – Graphs and Discrete Structures
November 24, 2021 – Lecture 8

1 graphs with no short cycles and large chromatic number

In this section we will see a way to prove the existence of graphs with no triangles or 4-cycles, but with large chromatic number. The downside is that the proof is only existential (we don't construct such graphs explicitly). Such graphs are, however, very difficult to construct explicitly. Moreover, the proof easily extends to the following: *for any g and k, there exists a graph of chromatic number at least k that has no cycles of length less than g.* The homework assignment for next week is to see how to modify the proof below to obtain such a result.

We will need the following simple probabilistic results.

Lemma 1 (Markov Inequality)**.** *If $X \geq 0$ is a discrete random variable, then for any $t > 0$, $\mathbb{P}(X \geq t) \leq \mathbb{E}(X)/t$.*

Proof. By definition,

$$\mathbb{E}(X) = \sum_x x\,\mathbb{P}(X = x) \geq \sum_{x \geq t} x\,\mathbb{P}(X = x) \geq t \sum_{x \geq t} \mathbb{P}(X = x) = t\,\mathbb{P}(X \geq t),$$

where the first inequality follows from the fact that $X \geq 0$. It follows that $\mathbb{P}(X \geq t) \leq \mathbb{E}(X)/t$, as desired. □

For integer random variables, Markov inequality has the following simple (yet very useful) consequence (just take $t = 1$).

Corollary 2. *If $X \geq 0$ is an integer random variable, then $\mathbb{P}(X \neq 0) \leq \mathbb{E}(X)$.*

To construct our triangle-free graphs of large chromatic number, we will need the notion of a *random graph*. For an integer n and a real $p \in [0, 1]$ (that might depend on n), let $G(n, p)$ be the (random) graph with n vertices, in which we add an edge with probability p, independently, between any pair of vertices.

Theorem 3. *For any k, there exists a graph with no triangles or 4-cycles and with chromatic number at least k.*

Proof. Fix an integer k. In order to prove that some graph G on n vertices has chromatic number at least k, it is enough to prove that $\alpha(G) \leq n/k$.

For some integer n, take $p = n^{-3/4}$ and consider the random graph $G(n, p)$. We will prove that with good probability, this graph has few triangles or 4-cycles, and its independence number is at most $\frac{n}{2k}$.

The expectation of the number of triangles is $\binom{n}{3}p^3 \leq n/100$ for large enough n. The expectation of the number of 4-cycles is $3\binom{n}{4}p^4 \leq n^4/8$. Thus by Markov inequality, the probability that the number of triangles plus the number of 4-cycles is larger than $n/2$ is at most $1/3$.

The expectation of the number of independent sets of size $n/2k$ is $\binom{n}{n/2k}(1-p)^{\binom{n/2k}{2}} \leq 2^n \exp(-pn^2/8k^2) = 2^n \exp(-n^{5/4}/8k^2)$, using that $1-p \leq \exp(-p)$. The last expression tends to 0 as n tends to infinity so when n is sufficiently large it is at most $1/3$. By Corollary 2, The probability that $G(n, p)$ contains an independent set of size at least $n/2k$ is at most $1/3$ (when n is sufficiently large).

It follows that when n is sufficiently large, the probability that $G(n, p)$ contains more than $n/2$ cycles of length at most 4 or an independent set of size $n/2k$ is at most $1/3 + 1/3 = 2/3$. So with probability at least $1 - 2/3 = 1/3$, $G(n, p)$ contains at most $n/2$ cycles of length at most 4, and no independent set of size $n/2k$. It follows that a graph G with such properties exists. In G, delete one vertex in each cycle of length at most 4, and let H be the resulting graph. We have deleted at most $n/2$ vertices, thus H has $n' \geq n/2$ vertices, and no cycle of length at most 4 (by construction). Moreover G (and thus H) contains no independent set of size $n/2k \leq n'/k$, and thus $\alpha(H) \leq n'/k$. Thus, $\chi(H) \geq n'/\alpha(H) \geq k$, as desired. $\square$

2 Crossing lemma and point-line incidences on the plane

Recall that a *planar graph* is a graph that can be drawn on the plane so that vertices are represented by points and each edge is represented by a not self-intersecting continuous curve connecting the corresponding points in such a way that no two curves corresponding to two edges intersect in an interior point of one of them. A *plane graph* is a planar graph together with some drawing satisfying the rules above. Euler's formula states that, if e, v, f are the numbers of the edges, vertices and faces in a drawing of a planar graph, respectively, then

$$v - e + f = 2.$$

Each face has at least 3 edges on the boundary, while each edge belongs to 2 faces, giving $2e \leq 3f$. Using this inequality, Euler's formula implies that

$$e \leq 3v - 6.$$

Thus, planar graphs have linearly many edges and so most of the graphs are very far from being planar in that sense. One way of quantifying how far a graph is from being planar is by introducing the crossing number $cr(G)$ of a graph G. For a drawing of G on the plane, we say that two edges *cross* if they intersect in an interior point of one of them. Then the *crossing number* $cr(G)$ of a graph G is the smallest number of pairs of crossing edges that a drawing of G on the plane can have.
Note that, by definition, $cr(G) \leq \binom{|E(G)|}{2}$. The following proposition gives a simple bound on $cr(G)$.

Proposition 4. *For any graph G on n vertices and with e edges we have $cr(G) > e - 3v$.*

Proof. Let us consider a drawing of G on the plane with $cr(G)$ crossings. Delete one edge from each crossing that this drawing has. If the number of crossings was at most $e - 3v$, then we deleted at most $e - 3v$ edges, and thus the resulting drawing (and the corresponding graph) has at least $3v$ edges. On the one hand, the drawing has no more crossings, and thus the corresponding graph is planar. On the other hand, it violates the consequence of Euler's formula displayed above. $\square$

Using probabilistic method, we can amplify this proposition to obtain an essentially best possible bound on the number of crossings of any graph.

Theorem 5 (Crossing Lemma)**.** *If G is a graph on n vertices and with e edges, where $e \geq 4n$, then*

$$cr(G) \geq \frac{e^3}{64n^2}.$$

Proof. Let us take a drawing of G with $cr(G)$ crossings. In such a drawing, no two edges starting from the same vertex cross. Indeed, if two curves f, g corresponding to two edges with a common endpoint v cross in a point x then we can redraw this part of the drawing as follows: replace f with f', where f' follows the trajectory of g up to x and then follows the trajectory of f; make a symmetric replacement g' for g; move g', f' slightly apart at x so that no other crossing is affected. Note that this procedure reduces the total number of crossings by one, which contradicts the minimality of the chosen drawing.
Next, let us construct a random subgraph H of G (and also a random sub-drawing of our drawing) as follows: include each vertex of G independently with probability p, where p is to be chosen later. Then the numbers of vertices $v(H)$ and edges $e(H)$ of H, as well as the number of crossings $c(H)$ in that particular drawing that we obtain are some random variables. Let us calculate their expectations. We have $E[v(H)] = np$, $E[e(H)] = ep^2$, $E[c(H)] = cr(G)p^4$. Note that in the last formula we used the fact that any crossing in our drawing occurs between edges with no common endpoints.
Next, let us apply Proposition 4 to H. We have $c(H) \geq e(H) - 3v(H)$ for any choice of H, and thus the same inequality holds for expectations:

$$E[c(H)] \geq E[e(H)] - 3E[v(H)].$$

Substituting the expressions for the expectations in this formula, we get

$$cr(G) \geq p^{-2}e - 3p^{-3}n.$$

Let us put $p = 4n/e$. Note that $p \in [0, 1]$ by our assumption on G. The displayed inequality then becomes

$$cr(G) \geq \frac{e^3}{16v^2} - \frac{3e^3}{64p^2} = \frac{e^3}{64v^2},$$

as required. □

Homework – Prove that the random graph $G(n, \frac{1}{2})$ contains no clique of size $2 \log n$ and no stable set of size $2 \log n$, with probability tending to 1 as $n \to \infty$.