ORCO – Graphs and Discrete Structures
December 1, 2021 – Lecture 9

We say that a point and a line are *incident* if the point lies on the line. We have seen before that in finite planes (i.e., finite projective planes) we can have as many as $n^{3/2}$ incidences between n points and n lines. It turns out that Euclidean plane is very different from finite planes in this respect, as is shown by the next theorem. The proof that we present is much simpler than the original proof and is an elegant application of the crossing lemma.

Theorem 1 (Szemerédi–Trotter)**.** *Given a set of n lines and n points in $\mathbb{R}^2$, the number of incidences between them is at most $5n^{4/3}$).*

Proof. Fix an arrangement of lines $\ell_1, \ldots, \ell_n$ and points $p_1, \ldots, p_n$ that maximizes the number of incidences between n points and n lines on the plane. Let α be the number of incidences in this arrangement. We construct a graph G with a drawing on the plane as follows. The vertices of the graph are simply $p_1, \ldots, p_n$, and we draw an edge (represented by a straight-line segment in the drawing) between the points that lie on the same line ℓ_i and are consecutive on this line. That is, if a line ℓ_i contains points $p_{i_1}, \ldots p_{i_k}$ which appear in that order on the line, then it corresponds to the edges $(p_{i_j}, p_{i_{j+1}})$, $1 \leq j \leq k-1$ in the graph, and to straight-line segments $p_{i_j}p_{i_{j+1}}$, $1 \leq j \leq k-1$ in the drawing. Note that the drawing that we get is 'contained' in the arrangement of lines $\ell_1, \ldots, \ell_n$ and that we simply throw away the rays emanating from the first/last point on the line.
The graph G has n vertices. Next, a line ℓ_i with k vertices contributes $k-1$ edges to G. This implies $|E(G)| = \alpha - n$. Finally, let us bound from above $cr(G)$. Clearly, it is at most the number of crossings in the given drawing of G. Next, each crossing in this drawing must correspond to the intersection of two lines ℓ_i, ℓ_j, and thus, knowing that any two lines intersect at most once, the total number of crossings is at most $\binom{n}{2}$.
The final step is to apply the crossing lemma and get the following chain of inequalities:

$$\frac{(\alpha - n)^3}{64n^2} \leq cr(G) \leq \binom{n}{2}.$$

Simplifying, we get that $\alpha - n \leq (64n^3(n-1)/2)^{1/3} \leq 4n^{4/3}$, and thus $\alpha \leq 4n^{4/3} + n \leq 5n^{4/3}$. We remark that, strictly speaking, the crossing lemma only applies for graphs H with $|E(H)| \geq 4|V(H)|$. However, if in our case we have $|E(G)| \leq 4n$, then $\alpha \leq 5n$, which is also fine. $\square$

The bound in this theorem is tight up to a constant factor, as we will show using an appropriate grid-based construction. Consider the set P of points (a, b), where $a \in \{1, \ldots, n\}$ and $b \in \{1, \ldots, 2n^2\}$ and a set L of lines $y = cx + d$, where $c \in \{1, \ldots, n\}$ and $d \in \{1, \ldots, n^2\}$. The total number of points is $2n^3$, the total number of lines is n^3, and it is easy to see that each line $\ell \in L$ is incident to exactly n points from P, that is, the total number of incidences is n^4.

1 Intersection theorems

In this section, we will discuss some classical results in extremal set theory and their geometric consequences.

Let $[n] = \{1, \ldots, n\}$ and 2^X stand for the set of all subsets of the set X. Also, let $\binom{X}{k}$ stand for the set of all k-element subsets of X. The type of questions we deal with here is as follows: how large can a family $\mathcal{F} \subset \binom{[n]}{k}$ be, given that for any pair $A, B \in \mathcal{F}$ $|A \cap B|$ avoids certain values.

The fist theorem of this type is the Erdős–Ko–Rado theorem. We say that a family $\mathcal{F}$ of sets is *intersecting* if for any two $A, B \in \mathcal{F}$ we have $|A \cap B| > 0$. Let us first show that if $\mathcal{F} \subset 2^{[n]}$ is intersecting, then $|\mathcal{F}| \le 2^{n-1}$, and this is tight. Indeed, to see the upper bound, note that out of each pair of sets $(X, [n] \setminus X)$ we can take only one set, and that the number of such pairs is 2^{n-1}. To see the lower bound, consider all sets containing 1. Actually, something much stronger holds, which we leave as an exercise.

Exercise: Show that any intersecting family $\mathcal{F} \subset 2^{[n]}$ is contained in another intersecting family $\mathcal{G} \subset 2^{[n]}$, where $|\mathcal{G}| = 2^{n-1}$.

Next, let us take a look at the situation for the families of k-element sets.

Theorem 2 (Erdős–Ko–Rado theorem)**.** *If $n \ge 2k > 0$ and $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, then $|\mathcal{F}| \le \binom{n-1}{k-1}$.*

Note that equality is attained on the family of all sets containing 1.

This theorem has many different proofs. We will present one that has a flavour of classical extremal set theory and uses a certain operation on families of sets, called *shifting*. For a given pair of indices $1 \le i < j \le n$ and a set $A \subset [n]$, define its (i, j)-*shift* $S_{ij}(A)$ as follows. If $i \in A$ or $j \notin A$, then $S_{ij}(A) = A$. If $j \in A, i \notin A$, then $S_{ij}(A) := (A - \{j\}) \cup \{i\}$. That is, $S_{ij}(A)$ is obtained from A by replacing j with i. The (i, j)-shift $S_{ij}(\mathcal{F})$ of a family

$\mathcal{F}$ is as follows:

$$S_{ij}(\mathcal{F}) := \{S_{ij}(A) \ : \ A \in \mathcal{F}\} \cup \{A \ : \ A, S_{ij}(A) \in \mathcal{F}\}.$$

We say that $\mathcal{A}$ is *shifted* if $S_{ij}(\mathcal{A}) = \mathcal{A}$ for any $1 \le i < j \le n$. Note that $\mathcal{F}$ is shifted if and only if for any $A = \{x_1, \ldots, x_k\} \in \mathcal{F}$ and any $B = \{y_1, \ldots, y_k\}$ such that $y_i \le x_i$, we have $B \in \mathcal{F}$. It is not difficult to see that any family can be made shifted after finitely many shifts: e.g., note that the total sum of all elements decreases by at least 1 as we perform a (non-trivial) shift S_{ij} with $i < j$, but this sum must clearly stay positive.
The following lemma has a straightforward, but slightly technical proof, and we leave it as an exercise.

Lemma 3. *If $\mathcal{F}$ is intersecting then $S_{ij}(\mathcal{F})$ is intersecting.*

Proof of the Erdős–Ko–Rado theorem. We prove the theorem by double induction on n, k. Specifically, we use the induction assumption for $(n-1, k-1)$ and $(n-1, k)$ to derive the statement for (n, k).
Let us check the base cases: $k = 1$ and $n = 2k$. The case $k = 1$ is trivial since $|\mathcal{F}| \le 1$ in this case. If $n = 2k$ then $\binom{n-1}{k-1} = \frac{1}{2}\binom{n}{k}$. We note that for each k-element set X its complement $\bar{X}$ is also k-element. As in the non-uniform case, we can include at most 1 out of each such pair in $\mathcal{F}$, getting the bound $|\mathcal{F}| \le \frac{1}{2}\binom{n}{k}$.
Next, assume $n > 2k > 2$. For the inductive step, we need the following notation:

$$\mathcal{F}(\bar{n}) := \{A \in \mathcal{F} : n \notin A\},$$
$$\mathcal{F}(\bar{n}) := \{A \setminus \{n\} : n \in A, A \in \mathcal{F}\}.$$

Using the lemma above, we may without loss of generality assume that $\mathcal{F}$ is shifted. Note that $\mathcal{F}(n) \subset \binom{[n-1]}{k-1}$, $\mathcal{F}(\bar{n}) \subset \binom{[n-1]}{k}$. Since $\mathcal{F}(\bar{n}) \subset \mathcal{F}$, we have that $\mathcal{F}(\bar{n})$ is intersecting. Using shiftedness, let us show that $\mathcal{F}(n)$ is intersecting as well. Indeed, if $A', B' \in \mathcal{F}(n)$ are disjoint then $A := A' \cup \{n\}$, $B := B' \cup \{n\}$ satisfy $A \cap B = \{n\}$ and, moreover, $A, B \in \mathcal{F}$. Since $|A \cup B| = 2k - 1 < n$, there is an element $x \in [n] \setminus (A \cup B)$. Using shiftedness, we conclude that the set $C := A' \cup \{x\}$ belongs to $\mathcal{F}$. But then $B \cap C = \emptyset$, a contradiction. This proves that $\mathcal{F}(n)$ is indeed intersecting.
Now we are ready to conclude the proof via the following chain. We use inductive assumption in the inequality below.

$$|\mathcal{F}| = |\mathcal{F}(n)| + |\mathcal{F}(\bar{n})| \le \binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1}. \qquad \square$$