ORCO – Graphs and Discrete Structures
December 8, 2021 – Lecture 10

1 Forbidden intersections and chromatic number of the space

It turns out that if we forbid one intersection 'in the middle', then we can get much stronger upper bounds on the size of the family. The following theorem is a beautiful illustration of the so-called linear-algebraic method in combinatorics. In this case our goal is to upper bound the size of a family. We are going to correspond a polynomial to each set and show that these polynomials are linearly independent. Then we are bounding the size of the family by the dimension of the space of polynomials that we used.

Theorem 1. *Let p be a prime number. Assume that $\mathcal{F} \subset \binom{[4p-1]}{2p-1}$ satisfies $|A \cap B| \neq p-1$ for any $A, B \in \mathcal{F}$. Then $|\mathcal{F}| \le \sum_{i=0}^{p-1} \binom{4p-1}{i}$.*

Note that, first, $\binom{[4p-1]}{2p-1} = 2^{(1+o(1)(4p-1)}$ and, second, $\sum_{i=0}^{p} \binom{4p-1}{i} < 1.8^{4p-1}$ for large enough p. The latter inequality is a standard calculation using Stirling formula, and we omit it. Also, recall that there is a prime number between n and $n + n^{2/3}$ for any sufficiently large n, and it's convenient to think that we have such a theorem for any sufficiently large n.

Proof. Put $n = 4p-1$ and consider such a family $\mathcal{F}$. To each set $A \in \mathcal{F}$, correspond a characteristic vector v_A of length n, where the i'th coordinate of v_A is either 0 or 1 depending on whether $i \in A$ or not. Next, correspond to A a polynomial

$$f_A(x) = \prod_{i=0}^{p-2} (i - \langle x, v_A \rangle),$$

where $f_A(x)$ is an n-variate polynomial over $\mathbb{F}_p$. In what follows, all the arithmetic is in $\mathbb{F}^p$. Note that $f_A(x)$ is a product of $p-2$ linear terms. Consider the following polynomial $g_A(x)$: open the brackets in the product defining $f_A(x)$, represent $f_A(x)$ as a sum of monomials, and replace each monomial $Cx_{i_1}^{\alpha_1} \cdot \ldots \cdot x_{i_s}^{\alpha_s}$ with $Cx_{i_1} \cdot \ldots \cdot x_{i_s}$. That is, we erase the powers of the variables in each monomial. The resulting $g_A(x)$ is multilinear and, importantly, we have $f_A(x) = g_A(x)$ for any $x \in \{0,1\}^n$. (This is because $0^\alpha = 0$ and $1^\alpha = 1$.)

We are going to show that g_A, when thought of as vectors in $\mathbb{F}_p^N$ for an appropriate dimension N, are linearly independent. Indeed, we first note the following two properties: (i) $g_A(v_A) = f_A(v_A) \neq 0$. The latter is true because $\langle v_A, v_A \rangle = |A| = p-1 \pmod{p}$ and thus no multiple in the definition of $f_A(v_A)$ is 0. (ii) We have $g_A(v_B) = f_A(v_B) = 0$ for $A \neq B$ for pretty much the same reasons. We have $\langle v_A, v_B \rangle = |A \cap B| \neq p-1 \pmod{p}$ (the value $2p-1$ is only possible for $A = B$, and value $p-1$ is forbidden), and thus there will be a multiple in $f_A(v_B)$ that is equal to 0.
Using these properties, it is not difficult to show that the polynomials are linearly independent. Indeed, assume that

$$\sum_{S \in \mathcal{F}} \beta_S g_S = 0.$$

Then, clearly, $\sum_S \beta_S g_S(x) = 0$ for any vector x. Substitute $x = v_A$ and note that $0 = \sum_S \beta_S g_S(v_S) = \beta_A g_A(v_A)$, and thus $\beta_A = 0$. This is true for any $A \in \mathcal{F}$, and thus the linear combination is trivial.
We conclude that $|\mathcal{F}| \leq D := \dim \operatorname{span}\{g_S : S \in \mathcal{F}\}$. We are only left to bound D. Each polynomial that we use is a multilinear polynomial of degree at most $p-1$, depending on a set of n variables. The natural basis in this space is the set of all possible monomials of degree at most $p-1$, and there are $\sum_{i=0}^{p-1} \binom{n}{i}$. The theorem is proved. $\square$

Exercise: show that, for a family $\mathcal{F} \subset 2^{[n]}$ such that (i) each size of the set in $\mathcal{F}$ is odd and (ii) intersection size of any two different sets is even, we have $|\mathcal{F}| \leq n$.

This result has a surprising application to an important problem in discrete geometry. Let $\chi(\mathbb{R}^n)$ stand for the chromatic number of the space, i.e., the minimum number of colors we need to color all points of the space so that no two points of the same color are at unit distance apart. It is not difficult to show that

$$4 \leq \chi(\mathbb{R}^2) \leq 7:$$

`https://en.wikipedia.org/wiki/Hadwiger-Nelson_problem`
For a while, the constructions in higher-dimensional spaces gave only polynomial lower bounds. The theorem above allows to show much more.

Proposition 2. *We have $\chi(\mathbb{R}^n) \geq 1.1^n$ for any sufficiently large n.*

Proof. We use notations as in the theorem above. First, consider the following graph $G = (V, E)$. We have $V = \binom{[4p-1]}{2p-1}$, and edges connect sets A, B if $|A \cap B| = p-1$. Then the Frankl–Wilson theorem, stated in the terms of this graph, says that $\alpha(G) \le \sum_{i=0}^{p-1} \binom{4p-1}{p-1}$. Using the remark after the theorem, we have $\chi(G) \ge \left(\frac{2}{1.8}\right)^{(1+o(1))(4p-1)}$.
Next, we find a geometric interpretation of G. Note that $v_A : A \in \binom{4p-1}{p-1}$ are points of the hypercube $\{0,1\}^{4p-1}$. We have $\|v_A - v_B\| = \|v_A\|^2 + \|v_B\|^2 - 2\langle v_A, v_B\rangle$, and, given that all vectors v_A have the same length, we note that if $\langle v_A, v_B\rangle = p-1$ then $\|v_A - v_B\| = r$, where r is independent of the choice of A, B. Thus, G can be realized in the space $\mathbb{R}^{4p-1}$ such that each vertex is a point in the standard hypercube and each edge in G corresponds to two points at distance r.
We note next that, using homothety, any proper coloring of $\mathbb{R}^n$ with 'forbidden distance 1' can be transformed into a proper coloring of $\mathbb{R}^n$ with 'forbidden distance r', and thus $\chi(\mathbb{R}^{4p-1}) \ge \chi(G) > \left(\frac{2}{1.8}\right)^{(1+o(1))(4p-1)}$. Finally, using the fact that prime numbers are dense, for any sufficiently large n we can choose p such that $n \ge 4p-1$ and $4p-1 = (1+o(1))n$ and thus

$$\chi(\mathbb{R}^n) \ge \chi(\mathbb{R}^{4p-1}) \ge \left(\frac{2}{1.8}\right)^{(1+o(1))(4p-1)} = \left(\frac{2}{1.8}\right)^{(1+o(1))n} > 1.1^n.$$

□

Let us briefly discuss the upper bounds on $\chi(\mathbb{R}^n)$.

Proposition 3. *We have* $\chi(\mathbb{R}^n) \le 9^n + 1$.

Proof. Let B_r stand for a ball with radius r and center in 0. For two sets $Y, Z \in \mathbb{R}^n$ we use notation $Y + Z = \{y + z : y \in Y, z \in Z\}$.
We say that a collection of balls form a packing if the balls do not intersect in their interior. We first construct a packing of closed balls of radius 1/2 in $\mathbb{R}^n$ greedily: in the order of increasing $\|x\|$, we check if we can add a ball with center in x in the packing. Let X be the resulting set of centers. Then note that the set $X + B_1$ covers the whole space. Indeed, if there is a point y not covered by the set, then $\|y - x\| > 1$ for any $x \in X$ and thus we could have added y to the 1/2-packing.
Next, fix $\epsilon > 0$ and let us construct a graph $G = (X, E)$, where we connect by edges any two points $x_1, x_2 \in X$ such that $\|x_1 - x_2\| \le 4 + 2\epsilon$. Let us bound the degree of a vertex x. The balls or radius 1/2 centered at y,

$y \in N_x$, are pairwise disjoint and all lie inside the ball $x + B_{9/2}$. given that $Vol(B_{9/2+2\epsilon})/Vol(B_{1/2} = (9+4\epsilon)^n$, we have that $|N_x| \leq (9+4\epsilon)^n$. Therefore, G is $(9+4\epsilon)^n$-degenerate, and we can properly color its vertices into $N = (9+4\epsilon)^n + 1$ colors.

Now let us transform this coloring $f : X \to [N]$ into the coloring of the space that avoids distance $2+\epsilon$. For that, simply color all points of $B_1 + x$ into the color $f(x)$. Then the distance between any two points inside $B_1 + x$ is at most 2. Moreover, for any x, y such that $f(x) = f(y)$ we have $\|x - y\| \geq 4 + 2\epsilon$, and thus using triangle inequality, the distance between any point in $B_1 + x$ and any point in $B_1 + y$ is at least $4 + 2\epsilon - 2 > 2 + \epsilon$. Therefore, it is a proper coloring of the space.

Finally, since we can take $\epsilon > 0$ arbitrarily small, we get the desired bound.

□