ORCO – Graphs and Discrete Structures
December 15, 2021 – Lecture 11

1 Chromatic number of Kneser graphs

In this lecture, let us consider the following graph $KG_{n,k}$. The vertices of $KG_{n,k}$ are all k-element subsets of $[n]$, i.e., $\binom{[n]}{k}$, and the edge set consists of all pairs of sets that are disjoint. In particular, $KG_{n,1}$ is a complete graph on n vertices and $KG_{5,2}$ is the famous Petersen graph `https://en.wikipedia.org/wiki/Petersen_graph`
Note that an independent set in $KG_{n,k}$ is a collection of pairwise intersecting k-element sets, and thus the Erdős–Ko–Rado theorem in these terms states that

$$\alpha(KG_{n,k}) = \binom{n-1}{k-1}.$$

Kneser asked for the chromatic number of $KG_{n,k}$. The graph has the following natural coloring using $n-2k+2$ colors. For each $i = 2k, \dots, n$ color in color i the sets whose maximal element is i. We have used $n-2k+1$ colors and the only subsets that are left uncolored are k-element subsets of $[2k-1]$. Any two of them intersect, and thus we use an extra color for them.
Kneser conjectured that this is best possible, i.e., that $\chi(KG_{n,k}) = n-2k+2$. This was proved by Lovász using topology.

Theorem 1 (Lovász)**.** *For any $n \geq 2k > 0$ we have $\chi(KG_{n,k}) = n-2k+2$.*

The proof that we present is rather simple, but, unexpectedly, it uses some geometry and topology. We need some preparations in order to spell it out.
Recall that a subset X of a metric space Y is *open* if for any $x \in X$ an ϵ-neiborhood of x is contained in Y. (Think of Y being the Euclidean space $\mathbb{R}^n$ or a sphere S^{n-1}.) A set X in Y is closed if for any converging sequence $x_1, \dots \in X$ its limiting point x is also in X. A complement of open set is closed and vice versa. A sphere S^{n-1} stands for the standard unit sphere in $\mathbb{R}^n$. A pair of points $x, -x$ on the sphere are called *antipodal*.
The main and only topological tool that we are going to use in this lecture is the following theorem of Lusternik-Schnirelman-Borsuk.

Theorem 2 (Lusternik-Schnirelman-Borsuk)**.** *If S^{d-1} is covered by d sets $C_1, \dots, C_d$, such that each of them is either open or closed, then there is i such that C_i contains two antipodal points.*

It is very instructive to think about the case $d = 2$. Note that without the assumptions on C_i it is easy to construct a covering of S^{d-1} with just 2 sets and no antipodal points: simply out of each pair of antipodal points $x, -x$ include the first one in C_1 and the second in C_2.
Next, we need a certain statement about points in general position.

Lemma 3. *For any integer N we can take N points on the sphere S^{d-1} so that no d points lie on a diametral hypersphere, i.e., a subsphere formed by intersecting S^{d-1} with a hyperplane passing through the center of S^{d-1}.*

If the points satisfy the requirement above then we say that they are in *general position*. Actually, in different situations different general position requirements are imposed, but the rule is that this is a property we get with probability 1 if we take points at random. Here, however, we will provide an explicit construction using a very useful object: the moment curve.

Proof. The *moment curve* is defined as follows: $\gamma(x) = (1, x, x^2, \ldots, x^{d-1})$. This is a curve in $\mathbb{R}^d$. Let us show that if we take any N points on this curve, then no d of these points lie on a hyperplane in $\mathbb{R}^d$ that passes through 0. A generic hyperplane passing through 0 has the form $c_1x_1 + \ldots + c_dx_d = 0$. Substituting here the point $\gamma(x)$, we get $c_1 + c_2x + \ldots + c_dx^{d-1} = 0$. This is a polynomial of degree at most $d-1$ and thus it has at most $d-1$ real root, so at most $d-1$ points from the moment curve can lie on any such plane.
All we are left to do is to replace each point v with αv for some constant $\alpha > 0$ so that αv is on S^{d-1}. The resulting points satisfy the requirement. $\square$

We are ready to prove the Lovász' theorem.

Proof. Fix any coloring of $KG_{n,k}$ into $n - 2k + 1$ colors and let us show that it is not proper. Take a map $f : [n] \to S^{n-2k+1}$ that maps $[n]$ into points in general position. For shorthand, we denote by $f(A)$ the image of the set $A \subset [n]$. Based on f and the coloring, we are going to construct a cover of S^{n-2k+1} by sets $C_0, \ldots, C_{n-2k+1}$. Namely, for each point $v \in S^{n-2k+1}$ consider the open hemisphere $S_v := \{x : \langle v, x\rangle > 0\}$. First, assume that S_v contains $f(A)$ for some $A \in \binom{[n]}{k}$. Let χ be the color of A in the coloring of $KG_{n,k}$, $\chi \in [n-2k+1]$. Then we include v into S_χ. (Note that we do this for each such set A and each color that appears.) If S_v does not contain the image of any $A \in \binom{[n]}{k}$, then we include S_v into C_0. Note that then S_v contains at most $k-1$ points $f(i)$, $i \in [n]$.

First, we claim that C_i, $i \in [n-2k+1]$, are open. Indeed, for each fixed set A of color i the set of all v such that S_v contains $f(A)$ is open. Second, C_i is the union of such sets over all A of color i, and the union of finitely many open sets is open. Next, $C := C_1 \cup \ldots \cup C_{n-2k+1}$ is open as well, and C_0 is simply the complement of C on the sphere, so it is closed.
Note that we have a cover of S^{n-2k+1} with $(n-2k+2)$ open or closed sets, and so we can apply the Lusternik–Schnirelman–Borsuk theorem, getting that for some i the set C_i contains antipodal points.
Assume first that $i > 0$. Then for some v S_v and S_{-v} contain $f(A), f(A')$, respectively, where the k-element sets A, A' are of the same color i. But S_v and S_{-v} are disjoint, so $f(A)$ and $f(A')$ are disjoint, and so A, A' are disjoint. This means that the coloring is not proper.
Next, assume that $i = 0$. But then for some v both S_v and S_{-v} contain at most $k-1$ points $f(i)$, i.e., there are at most $2k-2$ values of i so that $\langle v, f(i)\rangle \neq 0$. Thus, at least $n-2k+2$ points $f(i)$ lie on the diametral hypersphere $\{x : \langle v, x\rangle = 0\}$. But this contradicts our general position assumption. This completes the proof. $\square$