1 Posets

A partially ordered set (or poset) is a set $P$ together with an order $<$ on the elements of $P$. Recall that an order $<$ is a binary relation which is transitive ($u < v$ and $v < w$ imply $u < w$) and irreflexive ($u < v$ and $v < u$ cannot both hold). We write $u \leq v$ if $u < v$ or $u = v$. We say that $u$ and $v$ are comparable if $u \leq v$ or $v \leq u$, and incomparable otherwise.

A chain in a poset $(P, <)$ is a subset $S$ of $P$ such that any two elements of $S$ are comparable. An antichain is a subset $S$ of $P$ such that any two elements of $S$ are incomparable.

**Theorem 1** (Mirsky, 1971). The maximum size of a chain in a poset $(P, <)$ is equal to the minimum number of antichains in which the elements of $P$ can be partitioned.

*Proof.* For an element $u \in P$, let $s(u)$ be the maximum size of a chain which has $u$ as maximum element. For any integer $i \geq 1$, the set $s^{-1}(i) = \{u \in P \mid s(u) = i\}$ is an antichain. These antichains partition $P$, and their number is equal to the maximum size of a chain, as desired. ∎

Given a poset $(P, <)$, the comparability graph of the poset is the graph $G$ with vertex set $P$, and an edge between any two vertices that are comparable. Mirsky’s theorem immediately implies that comparability graphs are perfect (see Lecture 4 for the definition).

**Theorem 2** (Dilworth, 1950). The maximum size of an antichain in a poset $(P, <)$ is equal to the minimum number of chains in which the elements of $P$ can be partitioned.

*Proof.* We prove the result by induction on the number of elements in $P$. As the result is clear when $P$ is empty we can assume $|P| \geq 1$. Let $u$ be a maximal element of $P$ (in the sense that there is no element $v \in P$ with $u < v$), and let $P' = P \setminus \{u\}$. By the induction, $P'$ has a partition into $k$ chains $C_1, \ldots, C_k$ (for some integer $k$) and $P'$ also contains an antichain of size $k$. Note that any such antichain intersects each chain $C_i$ in exactly one
element. For \(1 \leq i \leq k\), let \(x_i\) be the maximal element of \(C_i\) that is contained in an antichain of size \(k\) in \(P'\), and let \(X = \{x_1, \ldots, x_k\}\).

Let us first prove that \(X\) is an antichain in \(P'\) (and thus in \(P\)). Consider any \(1 \leq i, j \leq k\), and assume for the sake of contradiction that \(x_i < x_j\). Look at an antichain \(Y\) of size \(k\) in \(P'\) containing \(x_j\). This antichain \(Y\) also intersects \(C_i\) in an element, call it \(y_i\). By maximality of \(x_i\), we have \(y_i \leq x_i\). So if we had \(x_i < x_j\), then by transitivity we would also have \(y_i < x_j\), contradicting that \(Y\) is an antichain. This shows that \(X\) is an antichain of size \(k\) in \(P'\) (and also in \(P\)).

Now, assume first that there is \(1 \leq i \leq k\) such that \(x_i < u\) in \(P\). Then \(C'_i := \{u\} \cup \{x \leq x_i \mid x \in C_i\}\) forms a chain in \(P\), and \(P \setminus C'_i\) has no antichain of size \(k\) by definition of \(x_i\). By the induction, \(P \setminus C'_i\) has a partition into \(k - 1\) chains, and by adding \(C'_i\) we obtain a partition of \(P\) into \(k\) chains, as desired (recall that we know that there is an antichain of size \(k\) in \(P'\), and thus in \(P\)).

Otherwise, we can assume that there is no \(1 \leq i \leq k\), such that \(x_i < u\). Then since \(u\) is a maximal element in \(P\), \(u\) in incomparable with the elements of \(X\) and thus \(\{u\} \cup X\) is an antichain of size \(k + 1\) in \(P\). Moreover, adding the singleton \(\{u\}\) to the partition \(C_1, \ldots, C_k\), we obtain a partition of \(P\) into \(k + 1\) chains, as desired.

Exercise. Prove that Dilworth’s theorem can be quickly deduced from König’s theorem (stating that the size of a maximum matching is equal to the size of a minimum vertex cover in any bipartite graph). Show also how to deduce König’s theorem from Dilworth’s theorem.

Given a poset \((P, <)\), the incomparability graph of the poset is the graph \(G\) with vertex set \(P\), and an edge between any two vertices that are incomparable. As before, Dilworth’s theorem immediately implies that incomparability graphs are perfect.

Given a set \(X\), we define the poset \(P_X\) of all subsets of \(X\) ordered by inclusion (\(P_X\) is more than just a poset, it is a distributive lattice). Note that for all sets \(X\) on \(n\) vertices, the resulting poset \(P_X\) is the same (up to isomorphism), so we will most of the time consider that \(X = [n] := \{1, \ldots, n\}\) without loss of generality.

For a set \(X\) on \(n\) elements, a chain \(S_1 \subset S_2 \subset \cdots \subset S_k\) in \(P_X\) is said to be symmetric if \(|S_1| + |S_k| = n\) and for any \(1 \leq i \leq n - 1\), \(|S_{i+1}| = |S_i| + 1\).
Theorem 3. The elements of $P_n$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ symmetric chains.

Proof. We first observe that any partition into symmetric chains has size $\binom{n}{\lfloor n/2 \rfloor}$, since each symmetric chain has to contain a different subset on $\lfloor n/2 \rfloor$ elements. To construct the partition, we proceed by induction on $n$. The result clearly holds for $n = 1$, so we can assume $n \geq 2$. Assume that the result holds for $n - 1$, that is $P_{n-1}$ has a partition into $r$ symmetric chains $C_1, \ldots, C_r$. Each chain $C_i$ gives rise to two chains in $P_n$: first we can look at the maximum set $S$ of $C_i$ and add a new set $S \cup \{n\}$ to the chain $C_i$. Second we can add the element $n$ to all sets of the chain and delete the last set $S$ from the chain. All these chains are clearly symmetric in $P_n$, and it is not difficult to see that they partition $P_n$. \qed