ORCO – Graphs and Discrete Structures
November 23, 2022 – Lecture 8

1 Posets

A *partially ordered set* (or *poset*) is a set P together with an order $<$ on the elements of P. Recall that an *order* $<$ is a binary relation which is *transitive* ($u < v$ and $v < w$ imply $u < w$) and *asymmetric* ($u < v$ and $v < u$ cannot both hold). We write $u \leq v$ if $u < v$ or $u = v$. We say that u and v are *comparable* if $u \leq v$ or $v \leq u$, and *incomparable* otherwise.

A *chain* in a poset $(P, <)$ is a subset S of P such that any two elements of S are comparable. An *antichain* is a subset S of P such that any two elements of S are incomparable.

Theorem 1 (Mirsky, 1971)**.** *The maximum size of a chain in a poset* $(P, <)$ *is equal to the minimum number of antichains in which the elements of* P *can be partitioned.*

Proof. For an element $u \in P$, let $s(u)$ be the maximum size of a chain which has u as maximum element. For any integer $i \geq 1$, the set $s^{-1}(i) = \{u \in P \mid s(u) = i\}$ is an antichain. These antichains partition P, and their number is equal to the maximum size of a chain, as desired. □

Given a poset $(P, <)$, the *comparability graph* of the poset is the graph G with vertex set P, and an edge between any two vertices that are comparable. Mirsky's theorem immediately implies that comparability graphs are perfect (see Lecture 4 for the definition).

Theorem 2 (Dilworth, 1950)**.** *The maximum size of an antichain in a poset* $(P, <)$ *is equal to the minimum number of chains in which the elements of* P *can be partitioned.*

Proof. We prove the result by induction on the number of elements in P. As the result is clear when P is empty we can assume $|P| \geq 1$. Let u be a maximal element of P (in the sense that there is no element $v \in P$ with $u < v$), and let $P' = P \setminus \{u\}$. By the induction, P' has a partition into k chains $C_1, \ldots, C_k$ (for some integer k) and P' also contains an antichain of size k. Note that any such antichain intersects each chain C_i in exactly one

element. For $1 \leq i \leq k$, let x_i be the maximal element of C_i that is contained in an antichain of size k in P', and let $X = \{x_1, \ldots, x_k\}$.
Let us first prove that X is an antichain in P' (and thus in P). Consider any $1 \leq i, j \leq k$, and assume for the sake of contradiction that $x_i < x_j$. Look at an antichain Y of size k in P' containing x_j. This antichain Y also intersects C_i in an element, call it y_i. By maximality of x_i, we have $y_i \leq x_i$. So if we had $x_i < x_j$, then by transitivity we would also have $y_i < x_j$, contradicting that Y is an antichain. This shows that X is an antichain of size k in P' (and also in P).
Now, assume first that there is $1 \leq i \leq k$ such that $x_i < u$ in P. Then $C_i' := \{u\} \cup \{x \leq x_i \mid x \in C_i\}$ forms a chain in P, and $P \setminus C_i'$ has no antichain of size k by definition of x_i. By the induction, $P \setminus C_i'$ has a partition into $k-1$ chains, and by adding C_i' we obtain a partition of P into k chains, as desired (recall that we know that there is an antichain of size k in P', and thus in P).
Otherwise, we can assume that there is no $1 \leq i \leq k$, such that $x_i < u$. Then since u is a maximal element in P, u in incomparable with the elements of X and thus $\{u\} \cup X$ is an antichain of size $k+1$ in P. Moreover, adding the singleton $\{u\}$ to the partition $C_1, \ldots, C_k$, we obtain a partition of P into $k+1$ chains, as desired. □

Exercise. Prove that Dilworth's theorem can be quickly deduced from Kőnig's theorem (stating that the size of a maximum matching is equal to the size of a minimum vertex cover in any bipartite graph). Show also how to deduce Kőnig's theorem from Dilworth's theorem.

Given a poset $(P, <)$, the *co-comparability graph* of the poset is the graph G with vertex set P, and an edge between any two vertices that are incomparable. As before, Dilworth's theorem immediately implies that co-comparability graphs are perfect (this can also be deduced from Mirsky's theorem above, together with a classical theorem of Lovász stating that complements of perfect graphs are also perfect).

Given a set X, we define the poset P_X of all subsets of X ordered by inclusion (P_X is more than just a poset, it is a distributive lattice). Note that for all sets X on n vertices, the resulting poset P_X is the same (up to isomorphism), so we will most of the time consider that $X = [n] := \{1, \ldots, n\}$ without loss of generality.

For a set X on n elements, a chain $S_1 \subset S_2 \subset \cdots \subset S_k$ in P_X is said to be *symmetric* if $|S_1| + |S_k| = n$ and for any $1 \leq i \leq n-1$, $|S_{i+1}| = |S_i| + 1$.

Theorem 3. *The elements of $P_{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ symmetric chains.*

Proof. We first observe that any partition into symmetric chains has size $\binom{n}{\lfloor n/2 \rfloor}$, since each symmetric chain has to contain a different subset on $\lfloor n/2 \rfloor$ elements. To construct the partition, we proceed by induction on n. The result clearly holds for $n = 1$, so we can assume $n \geq 2$. Assume that the result holds for $n-1$, that is $P_{[n-1]}$ has a partition into r symmetric chains $C_1, \ldots, C_r$. Each chain C_i gives rise to two chains in P_n: first we can look at the maximum set S of C_i and add a new set $S \cup \{n\}$ to the chain C_i. Second we can add the element n to all sets of the chain and delete the last set S from the chain. All these chains are clearly symmetric in $P_{[n]}$, and it is not difficult to see that they partition $P_{[n]}$. $\square$