ORCO – Graphs and Discrete Structures
September 28, 2022 – Lecture 1

1 Definitions

Given a graph G and an integer k, a *k-coloring* of G is an assignment of k colors (usually denoted by $\{1, \dots, k\}$) to the vertices of G such that any two adjacent vertices have different colors. Given a coloring, the set of all vertices with a given color is usually called *a color class* (and a coloring can be thought of as a partition of the vertex set into color classes).
The *chromatic number of* G, denoted by $\chi(G)$, is the least k such that G has a k-coloring.

If a graph G has a k-coloring we also say that G is *k-colorable*, and if $\chi(G) = k$ we also say that G is *k-chromatic*.

A *clique* in a graph G is a set of pairwise adjacent vertices in G. The *clique number* of G, denoted by $\omega(G)$, is the maximum number of vertices in a clique of G.
A related notion is that of a stable set. A *stable set* (or *independent set*) in a graph G is a set of pairwise non-adjacent vertices in G. Note that in a coloring of a graph G, each color class is a stable set.
Since in any coloring of G, all the vertices of a clique must have distinct colors, we have the following simple observation.

Observation 1. *For any graph G, $\omega(G) \leq \chi(G)$.*

It is easy to see that there exist graphs for which the inequality above is strict (for instance, odd cycles on at least 5 vertices). In the next section, we show how to construct graphs for which the difference between the chromatic and clique numbers is arbitrarily large.

Before that, let us study the class of 2-colorable graphs, also known as *bipartite graphs*. A 2-coloring is also called a *bipartition*. A classical result in graph theory is the following.

Theorem 2. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. Since odd cycles are 3-chromatic, any graph that contains an odd cycle has chromatic number at least 3, which proves the first direction. To prove the second direction, consider a graph G with no odd cycle. We can assume that G is connected (otherwise we consider each connected component separately). Fix a vertex r in G, and for each $i \geq 0$, define L_i as the set of vertices of G at distance exactly i from r (the distance between two vertices is the minimum number of edges on a path connecting the two vertices). Note that the sets L_i partition the vertex set of G. We now define a 2-coloring of G as follows: all the vertices of the sets L_i with i odd are assigned color 1, and all the vertices of the sets L_i with i even are assigned color 2.

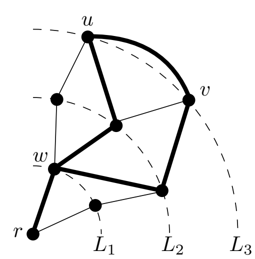

Figure 1: An illustration of the proof of Theorem 2.

We now prove that this is indeed a 2-coloring (assuming that G has no odd cycles). To this end, consider an edge uv of G (and assume by symmetry that the distance between r and u is at most the distance between r and v), and observe that by the definition of $(L_i)_{i\geq 0}$, either u and v both lie in some set L_i, or $u \in L_i$ and $v \in L_{i+1}$. In the second case, it follows from the definition of our coloring that u and v receive different colors. In the first case, consider a shortest path P_u between u and r, and a shortest path P_v between v and r. Let w be the vertex of $P_u \cap P_v$ that is the furthest from r (note that possibly $r = w$ if $P_u \cap P_v$ only consists of $\{r\}$). Now observe that the edge uv, together with the subpath of P_v between v and w, and the the subpath of P_u between u and w, forms an odd cycle (see Figure 1 for an illustration), which is a contradiction. □

It can be checked that the proof actually gives a polynomial algorithm to decide whether a graph is bipartite (and find a 2-coloring if this is the case, or an odd cycle otherwise). On the other hand, deciding whether the chromatic number of a graph is at most 3 is an NP-complete problem (even in very simple classes of graphs).

2 Mycielski's construction

We define a sequence $(M_k)_{k\geq 1}$ of graphs inductively. M_1 is a single vertex, and M_2 consists of two vertices joined by an edge. For $k \geq 3$, M_k is constructed as follows: we start with a copy of M_{k-1}, and for each vertex v in this copy of M_{k-1}, we add a vertex v' that has precisely the same neighbors as v (we say that v' is the twin of v). Finally, we add a vertex z^* that is adjacent to all the newly created vertices v', and non-adjacent to all the vertices of the copy of M_{k-1}.
It is not difficult to check that M_3 is a 5-cycle and M_4 is the so-called Mycielski graph, depicted below.

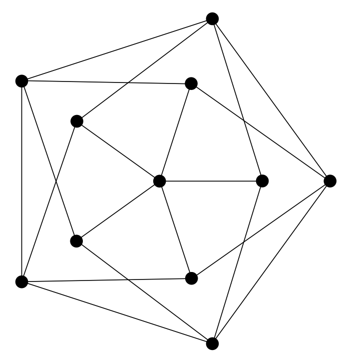

Figure 2: The graph M_4.

We now prove the following theorem.

Theorem 3. *For any $k \geq 1$, M_k is triangle-free (i.e. $\omega(G) \leq 2$) and $\chi(G) = k$.*

Proof. We prove the theorem by induction on k.
We start by proving that M_k has no triangle. This is clear if $k \leq 2$, so assume that $k \geq 3$. Let us denote by S the set of newly created vertices distinct from z^*. Assume for the sake of contradiction that there exist a triangle T in M_k. Since S is a stable set and M_{k-1} is triangle-free (by induction), T has two vertices in the copy of M_{k-1} (call them u, v) and one in S (call it w'). But since w' has the same neighbors in the copy of M_{k-1} as its twin w, uvw forms a triangle in M_{k-1}, which contradicts the induction hypothesis.

We now prove that for any $k \geq 3$, $\chi(M_k) = k$. The cases $k = 1$ and $k = 2$ are clear, so we can assume that $k \geq 3$. Since the copy of M_{k-1} is $(k-1)$-colorable (by induction), we can color it with colors $1, 2, \ldots, k-1$, then use color k for the vertices of S, and finally color 1 for z^*. This shows that $\chi(M_k) \leq k$. It remains to prove that $\chi(M_k) \geq k$. For this we will need the following simple observation.

For any graph H and any coloring of H with $\chi(H)$ colors, each color class contains a vertex that is adjacent to all the other color classes. (1)

To see why this holds, just observe that the negation of (1) implies that there is a color, say i, such that each vertex colored i is not adjacent to some other color class. In this case it is possible to recolor each vertex colored i with another color. But this results in a coloring of H with at most $\chi(H) - 1$ colors, which is impossible.

Now, assume for the sake of contradiction that M_k has a coloring with $k-1$ colors. Since $\chi(M_{k-1}) = k-1$ (by induction), we can apply (1) to the copy of M_{k-1} in M_k. This gives us sequence of vertices $v_1, v_2, \ldots, v_{k-1}$ in the copy of M_{k-1}, such that each v_i is colored i and is adjacent to all the other color classes. In particular this implies that for each i, the twin v_i' of v_i is also colored i. But then the vertex z^* is adjacent to vertices of colors $1, 2, \ldots, k-1$, which is a contradiction. □