1 Definitions

Given a graph $G$ and an integer $k$, a $k$-coloring of $G$ is an assignment of $k$ colors (usually denoted by $\{1, \ldots, k\}$) to the vertices of $G$ such that any two adjacent vertices have different colors. Given a coloring, the set of all vertices with a given color is usually called a color class (and a coloring can be thought of as a partition of the vertex set into color classes).

The chromatic number of $G$, denoted by $\chi(G)$, is the least $k$ such that $G$ has a $k$-coloring.

If a graph $G$ has a $k$-coloring we also say that $G$ is $k$-colorable, and if $\chi(G) = k$ we also say that $G$ is $k$-chromatic.

A clique in a graph $G$ is a set of pairwise adjacent vertices in $G$. The clique number of $G$, denoted by $\omega(G)$, is the maximum number of vertices in a clique of $G$.

A related notion is that of a stable set. A stable set (or independent set) in a graph $G$ is a set of pairwise non-adjacent vertices in $G$. Note that in a coloring of a graph $G$, each color class is a stable set.

Since in any coloring of $G$, all the vertices of a clique must have distinct colors, we have the following simple observation.

Observation 1. For any graph $G$, $\omega(G) \leq \chi(G)$.

It is easy to see that there exist graphs for which the inequality above is strict (for instance, odd cycles on at least 5 vertices). In the next section, we show how to construct graphs for which the difference between the chromatic and clique numbers is arbitrarily large.

Before that, let us study the class of 2-colorable graphs, also known as bipartite graphs. A 2-coloring is also called a bipartition. A classical result in graph theory is the following.

Theorem 2. A graph is bipartite if and only if it contains no odd cycles.

Proof. Since odd cycles are 3-chromatic, any graph that contains an odd cycle has chromatic number at least 3, which proves the first direction. To
prove the second direction, consider a graph $G$ with no odd cycle. We can assume that $G$ is connected (otherwise we consider each connected component separately). Fix a vertex $r$ in $G$, and for each $i \geq 0$, define $L_i$ as the set of vertices of $G$ at distance exactly $i$ from $r$ (the distance between two vertices is the minimum number of edges on a path connecting the two vertices). Note that the sets $L_i$ partition the vertex set of $G$. We now define a 2-coloring of $G$ as follows: all the vertices of the sets $L_i$ with $i$ odd are assigned color 1, and all the vertices of the sets $L_i$ with $i$ even are assigned color 2.

![Figure 1: An illustration of the proof of Theorem 2.](image)

We now prove that this is indeed a 2-coloring (assuming that $G$ has no odd cycles). To this end, consider an edge $uv$ of $G$ (and assume by symmetry that the distance between $r$ and $u$ is at most the distance between $r$ and $v$), and observe that by the definition of $(L_i)_{i \geq 0}$, either $u$ and $v$ both lie in some set $L_i$, or $u \in L_i$ and $v \in L_{i+1}$. In the second case, it follows from the definition of our coloring that $u$ and $v$ receive different colors. In the first case, consider a shortest path $P_u$ between $u$ and $r$, and a shortest path $P_v$ between $v$ and $r$. Let $w$ be the vertex of $P_u \cap P_v$ that is the furthest from $r$ (note that possibly $r = w$ if $P_u \cap P_v$ only consists of $\{r\}$). Now observe that the edge $uv$, together with the subpath of $P_v$ between $v$ and $w$, and the subpath of $P_u$ between $u$ and $w$, forms an odd cycle (see Figure 1 for an illustration), which is a contradiction. 

It can be checked that the proof actually gives a polynomial algorithm to decide whether a graph is bipartite (and find a 2-coloring if this is the case, or an odd cycle otherwise). On the other hand, deciding whether the chromatic number of a graph is at most 3 is an NP-complete problem (even in very simple classes of graphs).
2 Mycielski’s construction

We define a sequence \((M_k)_{k \geq 1}\) of graphs inductively. \(M_1\) is a single vertex, and \(M_2\) consists of two vertices joined by an edge. For \(k \geq 3\), \(M_k\) is constructed as follows: we start with a copy of \(M_{k-1}\), and for each vertex \(v\) in this copy of \(M_{k-1}\), we add a vertex \(v'\) that has precisely the same neighbors as \(v\) (we say that \(v'\) is the twin of \(v\)). Finally, we add a vertex \(z^*\) that is adjacent to all the newly created vertices \(v'\), and non-adjacent to all the vertices of the copy of \(M_{k-1}\).

It is not difficult to check that \(M_3\) is a 5-cycle and \(M_4\) is the so-called Mycielski graph, depicted below.

![Figure 2: The graph \(M_4\).](image)

We now prove the following theorem.

**Theorem 3.** For any \(k \geq 1\), \(M_k\) is triangle-free (i.e. \(\omega(G) \leq 2\)) and \(\chi(G) = k\).

**Proof.** We prove the theorem by induction on \(k\).

We start by proving that \(M_k\) has no triangle. This is clear if \(k \leq 2\), so assume that \(k \geq 3\). Let us denote by \(S\) the set of newly created vertices distinct from \(z^*\). Assume for the sake of contradiction that there exist a triangle \(T\) in \(M_k\). Since \(S\) is a stable set and \(M_{k-1}\) is triangle-free (by induction), \(T\) has two vertices in the copy of \(M_{k-1}\) (call them \(u, v\)) and one in \(S\) (call it \(w'\)). But since \(w'\) has the same neighbors in the copy of \(M_{k-1}\) as its twin \(w\), \(www\) forms a triangle in \(M_{k-1}\), which contradicts the induction hypothesis.
We now prove that for any $k \geq 3$, $\chi(M_k) = k$. The cases $k = 1$ and $k = 2$ are clear, so we can assume that $k \geq 3$. Since the copy of $M_{k-1}$ is $(k-1)$-colorable (by induction), we can color it with colors $1, 2, \ldots, k-1$, then use color $k$ for the vertices of $S$, and finally color 1 for $z^*$. This shows that $\chi(M_k) \leq k$. It remains to prove that $\chi(M_k) \geq k$. For this we will need the following simple observation.

For any graph $H$ and any coloring of $H$ with $\chi(H)$ colors, each color class contains a vertex that is adjacent to all the other color classes. (1)

To see why this holds, just observe that the negation of (1) implies that there is a color, say $i$, such that each vertex colored $i$ is not adjacent to some other color class. In this case it is possible to recolor each vertex colored $i$ with another color. But this results in a coloring of $H$ with at most $\chi(H) - 1$ colors, which is impossible.

Now, assume for the sake of contradiction that $M_k$ has a coloring with $k - 1$ colors. Since $\chi(M_{k-1}) = k - 1$ (by induction), we can apply (1) to the copy of $M_{k-1}$ in $M_k$. This gives us sequence of vertices $v_1, v_2, \ldots, v_{k-1}$ in the copy of $M_{k-1}$, such that each $v_i$ is colored $i$ and is adjacent to all the other color classes. In particular this implies that for each $i$, the twin $v'_i$ of $v_i$ is also colored $i$. But then the vertex $z^*$ is adjacent to vertices of colors $1, 2, \ldots, k-1$, which is a contradiction. \qed